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# Gauged N=1 Supergravity and the Embedding Tensor Formalism

**Jan De Rydt**

Dissertation presented in partial fulfillment of the requirements for the degree of Doctor of Science

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The Road Not Taken

Two roads diverged in a yellow wood,
And sorry I could not travel both
And be one traveler, long I stood
And looked down one as far as I could
To where it bent in the undergrowth;

Then took the other, as just as fair,
And having perhaps the better claim,
Because it was grassy and wanted wear;
Though as for that the passing there
Had worn them really about the same,

And both that morning equally lay
In leaves no step had trodden black.
Oh, I kept the first for another day!
Yet knowing how way leads on to way,
I doubted if I should ever come back.

I shall be telling this with a sigh
Somewhere ages and ages hence:
Two roads diverged in a wood, and I–
I took the one less traveled by,
And that has made all the difference.

*Robert Frost (1874-1963). Mountain Interval. 1920.*

# Preface

Gedurende acht jaar heb ik een persoonlijke weg doorheen het fysicalandschap bewandeld. Tijdens deze tocht kreeg ik de kans om de taal van de fysica te bestuderen en te gebruiken; zo werd het mogelijk om de vele facetten van deze fascinerende wereld beter te begrijpen. Zo kon ik ook zélf een klein stukje van dit boeiende universum in kaart te brengen.

Het eindresultaat van mijn tocht –of het reisverslag in zekere zin– ligt nu voor u. De fysicataal is alom tegenwoordig, en dat wordt duidelijk wanneer u dit boek op een willekeurige bladzijde openslaat. De vreemde woorden en symbolen zijn niet voor iedereen toegankelijk, en daarom wil ik niet nalaten om kort de contouren van dit werk te schetsen. Wie nadien graag meer wil lezen, verwijs ik naar de introductie (hoofdstuk 1) of de Nederlandse samenvatting (appendix B).

**In grote lijnen.** In deze thesis wordt onderzoekswerk voorgesteld dat kadert binnen het domein van de "hoge-energiefysica". Deze discipline bestudeert fysische fenomenen die zich zeer diep in de structuur van de materie manifesteren, meerbepaald op subatomair niveau. De hoofdrolspelers in die microscopisch kleine wereld zijn elementaire deeltjes, zoals het elektron of het foton. Al deze deeltjes interageren met elkaar via vier fundamentele interacties: de zwaartekracht, de elektromagnetische wisselwerking en de sterke en zwakke interacties. Voor elk van de *laatste drie* krachten hebben we een zeer nauwkeurige theoretische beschrijving. Bovendien voldoen ze alle drie aan dezelfde spelregels, namelijk die van een zogenaamde kwantumvelden theorie. Dit is een uniek kader waarbinnen we erg nauwkeurig het gedrag van de deeltjes kunnen berekenen en voorspellen. Maar deze beschrijving heeft ook een groot minpunt: de zwaartekracht past niet in dit plaatje. Als we toch proberen om zo'n kwantumveld voor gravitatie toe te voegen, stoten we onvermijdelijk op een aantal fundamentele problemen. Deze problemen zijn bij fysici al zeer lang bekend, maar pas vanaf de jaren tachtig werd ook echt vooruitgang geboekt in de zoektocht naar een oplossing. Een van de belangrijkste inzichten was dat de idee van elementaire puntdeeltjes moet worden opgegeven, en moet worden vervangen door het beeld van minuscule trillende snaartjes. Als we geen oog hebben voor de details, lijkt het alleen maar alsof deze snaartjes zich als puntdeeltjes gedragen. Maar er is meer. Een van de trillingstoestanden van de snaar komt precies overeen met het graviton, het deeltje dat verantwoordelijk is voor het overbrengen van de zwaartekracht! Snaartheorie biedt dus een consistent kader voor de beschrijving van alle deeltjes en hun onderlinge krachten, inclusief de zwaartekracht. In deze thesis zullen we bepaalde structurele aspecten van de snaartheorie onderzoeken, meerbepaald in een benadering waar de lengte van de snaar zeer klein wordt ten opzichte van de lengteschaal waarin we geïnteresseerd zijn. Dit regime wordt ook supergravitatie genoemd, en zoals de benaming suggereert is gravitatie een essentieel onderdeel van dit model. Waar we specifiek in geïnteresseerd zijn, is de beschrijving van de andere interacties (elektromagnetisme, enzovoort) binnen dit model.

**Samen op weg.** Heel wat mensen hebben bijgedragen tot het welslagen van dit avontuur. In de eerste plaats wil ik Toine bedanken, die me als ervaren gids heeft bijgestaan en heeft voorzien van de nodige bagage. Door zijn gedrevenheid en vertrouwen kreeg ik de kansen die van mij een betere wetenschapper hebben gemaakt. Many thanks also go to Marco, Jan, Torsten, Mario and Frederik for very fruitful collaborations. You have thought me about the importance of combined efforts in scientific research. I'm also grateful to the members of my jury, Dan, Désiré, Enrico, Marco and Walter, for carefully considering this manuscript and for judging my achievements in a positive and constructive way. It was a great honor to share passion, knowledge and friendship with so many people along the road. For numerous discussions about physics, life and everything else, and for many kind encouragements, I thank Alessandra, Bert, Bram, Brammmm!, Cedric, Davide, Dieter, Francesco, François, Frederik, Jan, Pantelis, Thomas, Walter and Wieland. For financing this journey, I thank my "sponsors", the FWO Vlaanderen and the Marie Curie Actions program of the European Union. Maar uiteindelijk was dit alles niet mogelijk geweest zonder de liefde, bezorgdheid, interesse en onvoorwaardelijke steun van mijn ouders, broer en zus. Pieter, Marieke, moeke en paps, acht jaar lang hebben jullie op onvervangbare wijze mijn basiskamp bemand. Elke beweging hebben jullie op de voet gevolgd, en telkens werd ik met open armen ontvangen. Daarom schuilt in deze thesis ook een stukje van jullie energie, gevoel en enthousiasme. Voor deze ongeschreven maar onmisbare bijdrage ben ik jullie allemaal heel erg dankbaar.

Jan De Rydt, Leuven, 8 september 2010

# Abstract

**English.** We report about our research on *gauged supergravities* with particular applications in four-dimensional $\mathcal{N} = 1$ theories. Part of this work was presented already in [1–3].

In the course of the last decade, high energy physics researchers have put a lot of effort in finding and understanding realistic low-energy solutions of string and M-theory. This has led to a revived interest in gauged supergravities, because these are the theories that naturally arise from string theory compactifications. Since gauged supergravities combine a (supersymmetric) description of gravity with the presence of a local internal symmetry group, they form an interesting class of supersymmetric low-energy effective models that describe all (known) forces in nature. In the light of future experiments that might put these theories to the test, and to improve our understanding of string and M-theory, it is important to classify all possible gauged supergravities and to investigate their properties.

In the first part of our work we use the symplectic structure of four-dimensional minimal supergravities to study the possibility of gauged axionic shift symmetries. This leads to the introduction of generalized Chern-Simons terms, and a Green-Schwarz cancellation mechanism for gauge anomalies. Similarly, we study the possibility of adding higher order derivative corrections to the two-derivative action, leading to a cancellation of the mixed gauge-gravitational anomalies. Our models constitute the supersymmetric framework for string compactifications with axionic shift symmetries, generalized Chern-Simons terms and quantum anomalies.

In the second part of this text we extend these results to a manifestly electric/magnetic duality covariant formalism. This formalism encompasses all possible gaugings in all possible duality frames via the introduction of magnetic vectors, two-forms and extra gauge transformations. Also generalized Chern-Simons terms have to be included from the start. The Green-Schwarz mechanism now involves anomalies that arise from the chiral coupling of fermions to the magnetic vectors. Our results are again relevant for string compactifications, especially in combination with background fluxes, which may naturally lead to actions with chiral fermionic spectra, tensor fields and gaugings in unusual duality

frames.

An important lesson we have learned is that the local gauge structure of the electric/magnetic covariant formalism is encoded in the so-called tensor hierarchy. Besides the two-forms, this hierarchy consists of higher order $p$-form fields and their gauge transformations. In the third and final part of this text we investigate the main properties of this gauge structure (up to two-forms) and we point out that the gauge algebra is soft, open and reducible. In order to deal with these complicated properties, we then motivate the use of a very suitable formulation in terms of the field-antifield (or Batalin-Vilkovisky) formalism. The latter provides us with a first step towards the quantization of general gauge theories.

**Nederlands.** In deze thesis bespreken we onderzoekswerk binnen het domein van *geijkte supergravitaties*, met als belangrijkste toepassing de vierdimensionale $\mathcal{N} = 1$ theorieën. Delen van dit werk werden reeds uitgebreid voorgesteld in de volgende publicaties: [1–3].

Gedurende de laatste tien jaar hebben onderzoekers heel wat tijd besteed aan het vinden en beter leren begrijpen van realistische lage-energie oplossingen van snaartheorie en M-theorie. De structuur van deze oplossingen wordt gegeven door zogenaamde "geijkte supergravitaties", die een supersymmetrische beschrijving van gravitatie combineren met het bestaan van lokale interne symmetrieën. Met het oog op toekomstige experimenten (zoals de LHC-CERN) en om onze kennis over snaar- en M-theorie te verbeteren, is het noodzakelijk om geijkte supergravitaties beter te begrijpen en om over een zo volledig mogelijke classificatie te beschikken.

In het eerste deel van dit werk gebruiken we de symplectische structuur van vierdimensionale supergravitatie om alle mogelijke (elektrische) ijkingen te onderzoeken. Zo wordt de mogelijkheid van geijkte "axion shift symmetrieën" in $\mathcal{N} = 1$ theorieën aangetoond, met de vereiste dat veralgemeende Chern-Simons termen aan de actie moeten worden toegevoegd. Bovendien laat deze constructie ook toe om ijkanomalieën uit de theorie te verwijderen – dit is een Green-Schwarz mechanisme. Terzelfdertijd kunnen correctietermen met een hoger aantal afgeleiden in de actie ervoor zorgen dat ook gemengde ijk-gravitatie anomalieën uit de theorie verdwijnen. Al deze modellen beschrijven een supersymmetrisch kader voor snaarcompactificaties met axion shift symmetrieën, veralgemeende Chern-Simons termen en een anomaal fermionisch spectrum.

In het tweede deel van deze tekst worden de hierboven genoemde resultaten uitgebreid naar een manifest elektrisch/magnetisch covariant formalisme. Dit formalisme laat ons toe om ijkingen in ongewone dualiteitsframes te bestuderen, en bovendien geeft het ons een manier om alle mogelijke ijkingen (zowel elektrisch als magnetisch) te classificeren. We zijn echter vooral geïnteresseerd in de implementatie van het Green-Schwarz mechanisme voor ijkanomalieën in dit formalisme. Het blijkt dat nu ook chirale koppelingen van de fermionen aan magnetische vectoren een rol spelen. Opnieuw zijn onze resultaten relevant voor snaarcompactificaties, in het bijzonder in combinatie met achtergrondfluxen die aanleiding geven tot acties met chirale fermionen, tensorvelden en magnetische ijkingen.

Een belangrijke les die we hebben geleerd is dat de lokale ijkstructuur van het elektrisch/magnetisch covariant formalisme gecodeerd is in een zogenaamde tensorhierarchie. Behalve de 2-vormen bevat deze hierarchie ook hogere orde $p$-vormen en hun ijktransformaties. In het derde en laatste deel van deze tekst onderzoeken we de belangrijkste eigenschappen van deze ijkstructuur (tot en met de 2-vormen) en we tonen aan dat de ijkalgebra soft, open en reducibel is. Om beter met deze ingewikkelde eigenschappen te kunnen werken, motiveren we tot slot het gebruik van een zeer geschikte formulering in termen van het veld-antiveld (of Batalin-Vilkovisky) formalisme. Dit formalisme is bovendien de eerste stap tot de kwantisatie van algemene ijktheorieën.

# Contents

CHAPTER 1

# INTRODUCTION

## 1.1 Effective theories

In everyday life we observe very diverse physical phenomena that have no obvious description within a single theoretical framework. Indeed, we use Kepler's laws to describe the motion of celestial objects, thermodynamics to study the properties of gases, Ohm's law to analyze electrical circuits, and so on. Although the very existence of these laws reveals a hidden systematics behind each of these phenomena, there is no obvious physical connection between them. For instance, it is not clear how the same physical principles can be responsible for both keeping the earth in its orbit around the sun, and describing the pressure in a gas of molecules. The only common properties of these laws are more formal: their validity is based on empirical grounds and their accuracy is restricted to a small set of macroscopical systems. Precisely due to this seemingly unconnected multitude of empirical laws, physicists have never considered them as the endpoint of their research. Instead, they wanted to find a deeper, microscopical explanation and at the same time, they were hoping to discover some universal order, i.e., a description that would give more structure to the diversity around us. Over the last 300 years, this urge for simplification and unification has slowly taken shape. Finally, during the 20th century, it has converged into the construction of two "fundamental" theories –general relativity and the Standard Model– that reduce the diversity in our universe to a remarkably small number of elementary matter particles and four interactions between them.

However, the knowledge of these elementary constituents and fundamental forces does not mean we do now understand all physical phenomena from first principles. For example, we are not capable of describing the interaction between protons and neutrons, starting from our mathematical model for the strong force between quarks. In turn, we are certainly not able to predict the behavior of molecules using the Standard Model, let alone we get any insight into unsolved problems such as the turbulent motion of a fluid that is made up by these molecules. In order to resolve this conundrum, physicists have always worked with so-called "effective models", models that remain ignorant about (part of) the details of the underlying microscopical structure.[1] All the empirical laws that we mentioned above are such effective models. For example, the Navier-Stokes equation does not require the knowledge of the forces between quarks that make up the atoms in a fluid, but it does (most certainly) describe turbulence properly. The parameter that determines how ignorant we can be about the underlying structure of our system is always related to an inherent physical scale of the system. For example, it is common practice to investigate the properties of atomic nuclei with experiments at the keV/MeV scale per nucleon and to compare the results with effective nuclear models, whereas elementary particles in the Standard Model are studied at much higher energy scales, up to TeV per nucleon at the Tevatron (Fermilab, Chicago) and the Large Hadron Collider (CERN, Geneva).

If we push this reasoning to its limit, the following natural question arises: how can we be sure that the Standard Model (SM) and Einstein's theory for gravity (GR) are not effective theories themselves? Of course, it is true that none of the experimental tests has found any contradiction between theory and experiment for either the SM or GR so far. Stronger yet, some parameters in the SM –such as the fine structure constant– are among the most accurately measured physical quantities to date, and there is a stringent correspondence to their calculated value. Therefore, we may conclude that within the limits of our current experimental setups, both the SM and GR work extremely well. However, it does not rule out the possibility of a physical regime beyond these limits, where these theories break down, i.e., where we cannot neglect a possible underlying structure. In fact, from theoretical arguments this is exactly what we expect. More precisely, at energies near the Planck scale ($\sim 10^{19}$GeV) and most certainly already below that scale, a new regime kicks in where the SM and GR lose their validity. Instead we have to consider a new and yet unknown theory, which theorists call "quantum gravity". But before we can grasp the need for such a theory, it is crucial to better understand the structure of the Standard Model and general relativity.

[1] We can be more precise here; we need two kinds of information in order to write down an effective theory for large-distance phenomena. First, we must know which parameters from the microscopic theory are relevant to large-distance physics. Second, we must know what degrees of freedom from the underlying theory appear at large distance scales.

## 1.2 Quantum gravity ... why do we care?

The Standard Model provides a unified description for the electromagnetic, weak and strong interactions within the framework of quantum field theory. Its basic ingredients are fields, including the electric and magnetic fields of 19th-century electrodynamics. Upon quantization, each of these fields corresponds to a quantum that represents a point-like object that we call an elementary particle. These particles interact with each other emitting and absorbing other particles; e.g. the electromagnetic force between electrons is mediated by the exchange of photons.

In order to deal more efficiently with emission and absorption processes in quantum field theory, Feynman proposed a nice diagrammatic representation of such particle interactions. The basic element is the local vertex, in which one particle disintegrates to yield two or more other particles. To each vertex one associates an object (a number or a matrix in general) that is proportional to the strength of the interaction between the particles that joint at this vertex. For example, when two electrons interact with a photon, the corresponding vertex is proportional to the electron charge, denoted by $e$. In general, this proportionality constant is called the coupling constant and it depends on the type of interaction and the energy scale of the process.[2]

Once the exact form of each vertex is known, quantum field theory provides an efficient algorithm to determine the probability for any physical process. It is obtained by forming diagrams in which vertices are connected in all possible ways.[3] To each diagram one associates a probability that is proportional to some power of the coupling constants involved in the process. The more internal vertices a diagram contains, the higher its power. In this way, quantum field theory provides an infinite series expansion for the outcome of any physical process, with the coupling constant as its expansion parameter. If, as for the electromagnetic interaction at low enough energies, the coupling constant is sufficiently small, one can truncate the power series at some point, and still make predictions about physical processes with very good accuracy. Such "perturbative calculations" are in most quantum field theories the only possible way to extract precise quantitative predictions.

However, we should point out that this is not yet the full story. Indeed, besides

[2] Usually the Lagrangian or the Hamiltonian of a system can be separated into a kinetic part and an interaction part. Then the coupling constant determines the strength of the interaction part with respect to the kinetic part, or between two contributions in the interaction part. In a quantum field theory, the energy dependence of the coupling constant is a quantum mechanical effect.

[3] In particular one should consider a sum over all physical processes between a given initial and final state. For each process with a fixed number of vertices, one must still integrate over all spacetime events at which interactions could have occurred, and integrate over the trajectories followed by the particles between the various vertices.

the successes of perturbative quantum field theory, it also exhibits certain present-day troubles in physics. First and above all, the above recipe to calculate the outcome of a physical process from a power series of Feynman diagrams, often results into an infinite result. Not only the series itself can be divergent, but we find infinities at each order. This difficulty was first overcome in quantum electrodynamics through a process called renormalization. It removed the infinities in an unambiguous way, and the finite predictions could then be compared with experiment. This procedure is only possible for a special class of quantum field theories where the infinities can be fully compensated for by corrections on the basic parameters of the theory, such as the mass and the charge of the electron. In a nutshell, this can be understood as follows. The observed electron mass is the sum of two contributions; the "bare mass" which appears as a parameter in the Lagrangian, and the "self-energy" resulting from the interaction of the electron with its own electromagnetic field. Only the sum of the two terms is observable. The self-energy can be calculated and turns out to be infinite. Nothing is known about the bare mass, and so it can be assigned a negatively infinite value, with the condition that the two infinities cancel and yield the observed finite mass of the electron. The same procedure can be applied to the other parameters in the theory, and we call a theory "renormalizable" if all infinities can be absorbed into a redefinition of the parameters. It turns out that the Standard Model is such a renormalizable theory and therefore it is perfectly capable of making finite predictions for all physical processes, involving only a finite number of physical parameters (roughly eighteen) that have to be determined experimentally.

The story for gravity, on the other hand, is quite different. Of the fundamental forces in nature, gravitation was the first for which an accurate mathematical description was found (Newton, 1687). Later, Newton's laws were reconciled with the principles of special relativity in Einstein's theory of general relativity. It resulted into a more accurate description of gravity in the regime where velocities are high (comparable to the speed of light) and masses are large. But above all, we should note that GR is a classical theory that does not play a role at the energy scales where the SM is important. Indeed, the gravitational interaction between an electron and a proton is $10^{37}$ times weaker than the electromagnetic force between these two particles[4], at least at energies that are low enough. However, if we extrapolate this comparison to higher interaction energies, we come to a remarkable conclusion. First, we observe that all interaction strengths of the Standard Model forces (electromagnetic, weak and strong) become roughly equal to one another at an energy of $10^{16}$GeV, and the gravitational force has the same strength at the Planck scale, being $10^{19}$GeV. Therefore, at such high energies (or small distances), gravity cannot be neglected and one needs a quantum description for this force. However, a straightforward quantization leads to one serious complication: gravity is not renormalizable. It means that the infinities in the

[4] Although gravitation is extremely weak, it still determines the large-scale structure of the universe since it is the only force that is both long-range and attractive between all particles.

Feynman diagrams involving gravitons (the particles that carry the gravitational force) are too severe to be removed. The reason for this is quite simple; the strength of the gravitational force increases when the energies of the virtual particles in the Feynman diagrams increase. Therefore, if one sums over all possible energies of the virtual particles, the higher energies give a larger contribution and lead to more serious infinities.[5] The presence of these infinities is inconvenient since one cannot make exact predictions about the outcome of an arbitrary physical process in terms of a finite number of measurable parameters. However, in contrast to what physicists thought in the beginning, these infinities are not problematic and they do certainly not mean that GR is badly flawed. Instead, they warn us that we are trying to push Einstein's theory beyond its limits of validity. In the end, we should look at GR as a highly-suppressed, non-renormalizable interaction that works fine at low energies (compared to $m_{\text{Planck}}$), but that cannot be used for energies of order $m_{\text{Planck}}$ because it is ignorant about the physics at these energies. The full description of physics at the Planck scale is instead provided by some yet to be constructed theory of quantum gravity, with Einstein gravity as its low-energy limit.

## 1.3 Is nature supersymmetric?

Over the last 30 years, theoretical physicists have obtained a good idea of how such a theory of quantum gravity might look like. There even exists a very serious candidate: superstring theory. According to this theory, elementary particles are not idealized points, but objects extended along one dimension (a string) or membranes with more dimensions. Strings interact with each other by joining and splitting, with an interaction strength that is controlled by the string coupling constant $g_s$. Since these interactions are "smeared out" due to the finite string size, string theory is finite order by order in perturbation theory and therefore it does not suffer from the infinities in ordinary quantum field theories.

However, the hope to ever observe the extended stringy objects is almost non-existing, since that would require an accelerator that reaches energies near the Planck scale, which is $10^{15}$ times higher than what is currently achievable with the LHC. So one might wonder whether we will ever be able to put string theory to the test. Luckily, though, one expects some properties of string theory to penetrate all the way down to lower energies, where they can be observed. One of these ingredients that might become observable in the future, is the presence of a new space-time symmetry, called supersymmetry.[6] The remarkable fact about

[5] Recall that renormalizable interactions such as electromagnetism only depend on the charges of the particles and not on the masses or energies.

[6] Supersymmetry does not only play a crucial role in constructing well-behaved string theories. It also provides a satisfactory explanation for the weak hierarchy $m_{\text{weak}} << m_{\text{Planck}}$, it yields an improved gauge coupling unification at the GUT scale and it naturally produces dark matter

supersymmetry is that it predicts the presence of new particles at low energies (these particles are "superpartners" of the known bosons and fermions). Indeed, the main property of supersymmetry is its tendency to pair up fermions, such as the quarks and leptons that make up ordinary matter, with bosons, such as the photon or the graviton. Therefore, we expect to find e.g. new bosonic partners of the quarks, called squarks, or a fermionic partner of the graviton, called the gravitino, etc. The reason why these new particles have not been observed yet, is because supersymmetry is broken at low energy, and therefore these particles acquire a mass that is beyond the reach of today's particle accelerators. However, physicists hope that the LHC will be able to probe the regime where superpartners become 'visible'. From theoretical arguments –such as the hierarchy problem– this is expected to happen near the TeV scale.

Independently of string theory, the possibility of supersymmetry as a four-dimensional symmetry was already considered by Wess and Zumino in 1974 [5, 6], in the context of ordinary quantum field theories. They constructed the first supersymmetric extensions of these theories upon introducing appropriate superpartners for the original (bosonic) fields. These models provided a surprisingly elegant way to describe both bosonic and fermionic degrees of freedom within one theoretical framework. Soon after this success, Ferrara, Freedman and Van Nieuwenhuizen also found a way to include gravity [7], and realized that supersymmetry had to be a local symmetry, meaning that all space-time points are transformed independently. Such supersymmetric theories of gravity –called supergravities– looked very promising in the beginning, because they provided a restrictive (and yet sufficiently broad) framework that could encompass both the known Standard Model interactions and gravity. Before people started to take string theory seriously, supergravity theories were even thought to be valid theories for quantum gravity because early quantum calculations showed that many divergent Feynman diagrams that appeared in general relativity were now canceled by new contributions that involved the propagation of supersymmetric partners. Therefore, it appeared as if supergravity was finite, at least up to second order in the diagrammatic expansion. However, later calculations showed that divergences were present at higher orders, and that the most interesting supergravity models –which have sufficient freedom to also include the Standard Model– are non-renormalizable all together.

This setback did not signal the end of the supergravity-era, though. Instead, when string theory became popular by 1984, the perspective on supergravity shifted from being a (failed) quantum gravity candidate to being a low-energy relic of string theory itself. Indeed, in the long-distance limit, when all finite string size effects can be neglected, string theory is effectively described by supergravity. Another

candidates. Moreover, under some reasonable assumptions, supersymmetry is the only possible extension of Poincaré invariance to a larger spacetime symmetry. In fact, this is how it got identified by Haag, Łopuszanski and Sohnius [4].

way to phrase this connection is in terms of the physical degrees of freedom. If we consider the spectrum of a quantized closed superstring, it contains a massless spin-2 excitation that can be identified as the graviton. Furthermore, the massless spectrum[7] of open and closed strings contains spin-1 fields, as well as scalar fields and antisymmetric $p$-forms in general. The latter can be seen as generalizations of the ordinary Maxwell fields in electromagnetism. The dynamics of all these massless fields, as determined by string theory, is governed by a supergravity action that contains a kinetic term for each propagating particle (including a Einstein-Hilbert term for the graviton), Standard Model-like interactions involving spin-1 fields, and numerous other couplings whose structure is strongly restricted by diffeomorphism and supersymmetry invariance. Due to this close connection between string theory and supergravity, a lot of the present-day advances in string theory are actually achieved through supergravity considerations.

## 1.4 Gauging reality

There is one aspect, however, that we have overlooked so far, which is the rather disturbing fact that string theory naturally lives in ten space-time dimensions. This observation leads to the conceptual problem of connecting the underlying ten-dimensional structure to a more familiar (and observable) four-dimensional theory. A precise understanding of how this connection works is of vital importance if we ever want to use string theory to make predictions (or "postdictions") about unknown (or unexplained) observable phenomena. However, as we will see in due course, there does not yet exist an unambiguous way to overcome this problem.

The standard procedure to tackle the six extra dimensions in string theory is to study configurations on a ten-dimensional space-time of the form $\mathbb{R}^{3,1} \times K^6$, with $\mathbb{R}^{3,1}$ the four-dimensional Minkowski space and $K^6$ some compact internal manifold. Various other ingredients can be added to this basic setup (such as fluxes, D$p$-branes and/or orientifolds ), but the fundamental idea always remains the same: if the compact dimensions are small enough, these solutions have an effective long-distance description in terms of some four-dimensional supergravity theory. This procedure is called "dimensional reduction" and it provides a very natural and satisfying explanation for our perception of a four-dimensional world. However, this conceivable method also produces one of the major present-day challenges that string theory has to cope with. Over the years, an enormous amount of consistent string theory reductions have been constructed and the details of the corresponding four-dimensional theories strongly depend on properties such as the shape of $K^6$ or the type of fluxes involved. However, string

[7]In addition to the massless fields, there is also a tower of massive string states with masses that are of the order of the inverse string length, $1/l_s$. These masses are naturally taken to be of the order $10^{18}$GeV, although a lower string scale is not excluded.

theory has not yet provided us with a mechanism to select out of this enormous amount of possibilities the (unique) solution that describes our universe. Therefore it is very hard, if not impossible, to make any definite physical predictions at all.

So how should we proceed? Is there any choice of compactification at all that yields the physical properties that we observe? Would it be sensible to exhaust all (known) possibilities and see whether any of these match reality? None of these questions seem to have an easy answer. However, string theorists have come a long way to making them more precise, and even, in some cases, answering them. For example, it is known how to link the existence of three quark and lepton generations to the geometry of the internal manifold $K^6$. Other properties of the force and matter particles, such as their mass, can also be explained by the geometry of the extra dimensions. The charges and types of interactions (electromagnetic, weak, etc.) are connected to the fluxes that are turned on, and so on. Although the details of these constructions are difficult to follow, let it be clear that string theory does indeed provide a framework to answer questions –such as why electrons and other particles have the masses they do– that we could not address in previous theories such as the Standard Model. However, we are still far from having all the answers, especially because there is no precise mechanism to select the "right" string configuration from the infinite set of possibilities.

As long as such a set of selection rules does not exist, we are forced to look for generic, rather than specific, aspects of a universe consisting of strings. Generic in this context refers to the universal properties that string theory imposes on us, independent of the details of the compactification. Such properties can be safely discussed, even without a complete understanding of the full theory. Two of these fundamental characteristics have been mentioned before: (i) string theory predicts the existence of a graviton whose (long-distance) interactions are those of general relativity, and (ii) string theory imposes supersymmetry as a new symmetry of space-time (at least at a certain energy scale). In four dimensions, these properties are nicely combined in a theory of pure supergravity, describing a graviton and, depending on the amount of supersymmetry, a number of superpartners (gravitini, scalars, etc). Besides these basic ingredients, there are two important extensions that we have briefly touched upon earlier in this text, but that require some more attention.

- String theory predicts the presence of extra matter fields on top of the universal graviton, gravitini, etc. For example, the minimal field content in four dimensions can be extended by an arbitrary number of spin-1/spin-$\frac{1}{2}$ and spin-$\frac{1}{2}$/spin-0 doublets. In principle the properties of these extra fields depend on the internal geometry of the compactification, but they can also be studied independently in a supergravity context. The advantage of these "matter-coupled supergravities" is that the number of particles can be adjusted almost at will in order to match the observed spectrum of elementary particles.

- String theory compactifications generically produce interactions between spin-1 fields and other matter particles, whose structure is similar to that of the Standard Model forces. This feature provides various new and interesting perspectives. First of all, it gives an explanation as to why all fundamental interactions have the same universal structure involving the exchange of spin-1 vector particles. Secondly, it is possible to embed the Standard Model forces into a supergravity theory. And thirdly, it opens the option of weak, long-range force fields that have gone undetected so far (so-called hidden sectors). These new interactions could clarify certain hypothetical processes that are not allowed by the Standard Model, such as a slow proton decay or possible transmutations of various combinations of quarks. The supergravity theories that include such interactions are called "gauged supergravities".

Given all these characteristics that are fundamental to string theory, we hope to have convinced the reader that it is worthwhile studying general four-dimensional supergravity theories. Since they constitute the most versatile testing ground for string theory characteristics that might show up in future experiments, it is of the outmost importance to understand their structure and to establish an exhaustive classification of all possible matter couplings and gaugings. In the end, this classification can be used for the construction of realistic models in close interaction with experimental observations. Our work in this thesis should be seen in this light.

## 1.5 Topics studied in this thesis

The main theories of interest in this thesis are four-dimensional supergravities with a minimal amount of supersymmetry (conventionally denoted by $\mathcal{N} = 1$) and general couplings to vector (spin-1 and spin-$\frac{1}{2}$) and chiral (spin-$\frac{1}{2}$ and spin-0) doublets. The structure of these theories was intensely studied in the past [8, 9], but there were still several open problems. In the first place, a complete catalogue of all gauged versions did not yet exist. In our work we take an important step forward towards that goal, and we unravel the intricate (gauge) structure of these theories. Secondly, we find an intricate interplay between the gaugings and certain first-order quantum aspects of the theory. More precisely, we obtain the general cancellation conditions for quantum anomalies, using a Green-Schwarz mechanism. Finally, we also take a first step towards the quantization of gauged supergravities via an embedding into the field-antifield formalism. Let us briefly summarize the purpose and content of each chapter, and provide some more details about the actual results that we have obtained.

The first three chapters of this thesis are devoted to the introduction of necessary background material. The flow of these developments is very linear and is meant to be accessible to all readers with a limited knowledge of gauge theories (chapter 2), supersymmetry and supergravity (chapter 3) and/or gauged supergravities (chapter 4).

Because all results in this thesis are in some way related to the physical principle of gauge invariance, *chapter 2* contains a gradual approach towards the general structure of gauge theories. As we noted above, the prototypical example of a gauge theory is the Standard Model, whose structure is completely fixed by the underlying $U(1) \times SU(2) \times SU(3)$ gauge group. In the first part of chapter 2, we will mainly use electromagnetism (the $U(1)$-part in the above gauge group) as an example to explain the logic behind gauge invariance, and to introduce the most important definitions such as gauge fields, field strengths, etc. Following this basic introduction, a general procedure will be developed that will allow us to construct more complicated non-Abelian (Yang-Mills) gauge theories. These include e.g. the weak and strong interactions in the Standard Model, which have a non-Abelian $SU(2)$ and $SU(3)$ gauge group respectively. This constructive method, called the gauging procedure[8], turns out to be essential for the later development of gauged supergravities. We end the first chapter with a brief introduction to the concept of gauge anomalies. These are first-order *quantum effects* that signal the violation of classical gauge invariance by quantum corrections. We will discuss how these anomalies arise, what their precise form is, why they are relevant and how we have to deal with them. The main conclusion is that gauge anomalies render the quantum theory inconsistent, and therefore we have to avoid their presence. In the Standard Model, such anomalies are automatically absent due to a miraculous interplay between the gauge structure and matter fields. But for more general gauge theories such as the gauged supergravities we will discuss later, anomaly freedom is not immediate and certain consistency conditions will have to be imposed. The derivation of these conditions will be part of our work in chapters 5 and 6.

In *chapter 3* we continue to study interacting quantum field theories, but we shift our attention from their gauged internal symmetries towards their space-time symmetries. First, we present a less familiar perspective on diffeomorphism invariance and general relativity, namely, we introduce Einstein's theory as a gauge theory invariant under local translations and local Lorentz transformations.

[8] In a nutshell, the gauging procedure consists of the following basic steps. One starts from a non-interacting quantum field theory whose dynamics is invariant under some global *internal* symmetry. For example, the free Dirac theory is invariant under global $U(1)$ phase transformations of the fermion. In the second step, one promotes these global transformations to *local* (i.e., space-time dependent) transformations. The original free theory is only invariant under these local variations –called gauge transformations– if an appropriate set of interactions is added. In our example, in order to obtain a $U(1)$ gauge invariant theory, the Dirac fermion has to be charged under the electromagnetic field.

The corresponding gauge fields –being the graviton and the spin connection– will be introduced, and we construct a kinetic action for these fields, which corresponds to the familiar Einstein-Hilbert term. The same gauging procedure will then be applied to supersymmetry. First we discuss the properties of globally supersymmetric theories, with an emphasis on four dimensions and $\mathcal{N} = 1$. The super-Poincaré algebra will be introduced, as well as its multiplet representations containing bosons and fermions. Also some insight will be given into the general structure of supersymmetric actions for these fields. Once the global properties of supersymmetry are well understood, we proceed to make this symmetry local. If one works through the details of this procedure, one automatically and inevitably has to introduce gravity. The resulting theories precisely correspond to the basic supergravities that we have discussed earlier in this introduction.

The next step in our discussion is the construction of four-dimensional *gauged* supergravities. This will be the main subject of *chapter 4*. We present an exhaustive review of the results that have already been obtained in the literature, whereas a presentation of our own results in this context will be given in chapters 5-7. The general strategy that is used for the construction of gauged supergravities is based on our results in chapter 2 about the gauging procedure. One starts from a basic $\mathcal{N} = 1$ supergravity coupled to non-interacting vector and chiral multiplets, and one studies the global internal symmetries that leave the dynamics of this free theory invariant. Generically, the global symmetry group turns out to be rather large and it has some intriguing properties. For example, it contains a generalization of ordinary electromagnetic duality which tells us that electric and magnetic phenomena cannot be distinguished in the absence of sources.[9] Once the global symmetry group of the non-interacting theory is identified, we will investigate which of these symmetries can be promoted to *local* invariances of a new, *interacting* theory. In other words, we will apply the gauging procedure that was discussed in chapter 2. The only allowed interaction types are gaugings with a minimal coupling to the (electric) vector fields. Their form depends on (i) the precise details of the global symmetry group, (ii) the amount of spin-1 fields that are available for gauging, and (iii) the partition into electric and magnetic vector fields (i.e., the choice of electromagnetic duality frame). Because the matter fields are electrically charged after the gauging, electromagnetic duality invariance is explicitly broken. Moreover, the admissible gaugings depend on the choice of duality frame. This adds an extra complication to the classification of *all* gaugings, because one has to scan all possible frames. In order to deal with these inconveniences, a much more powerful gauging procedure will be introduced, which is called the embedding tensor formalism. Its greatest virtues are manifest electromagnetic duality covariance (no frame-choice is required) and the capability to classify all possible gaugings using simple group theoretical arguments. The downside is the complicated structure of fields and transformations; in addition

[9] In other words, Maxwell's equations in vacuum are invariant under the exchange of electric and magnetic fields.

to the usual electric gauge fields and corresponding gauge parameters, one has to introduce magnetic gauge fields, antisymmetric 2-forms and a variety of new local transformations. This is the prize one has to pay for a duality covariant formulation.

With the recapitulation of these well known results, we have paved the way for further developments. In the final three chapters of this thesis, our research results will be presented, roughly corresponding to the three publications [1], [2] and [3].

In *chapter 5* the known results for electric gaugings will be generalized in two directions, inspired by recent progress in string theory compactifications [10–12]. First, a larger part of the electromagnetic duality transformations will be gauged, namely the subgroup that works on the scalars with a shift symmetry. Under these local shift transformations, the original $\mathcal{N} = 1$ supergravity action is not invariant anymore, because the Peccei-Quinn term gives rise to a non-vanishing contribution. In order to restore gauge (and supersymmetry) invariance, we have to add generalized Chern-Simons terms to the action.[10] These terms are cubic and quartic in the gauge fields, and their transformations exactly cancel the non-trivial contributions from the Peccei-Quinn term. Still, it is not possible to gauge all shift symmetries in this way, because gauge and supersymmetry invariance impose certain conditions on the generators. However, in the second part of chapter 5, we will show how these conditions can also be lifted. Indeed, if we relax the constraints on the local shift transformations of the scalars, the classical action is not supersymmetric and gauge invariant anymore, but its non-trivial variations can now be used to cancel quantum anomalies due to the chiral couplings in the theory. This is the Green-Schwarz mechanism. In particular, we will discuss the necessary and sufficient conditions for the cancellation of gauge, supersymmetry and mixed anomalies. All these results are interesting for rather different applications, ranging from orientifold compactifications with anomalous fermion spectra, to phenomenological models with $Z'$ bosons and manifest supersymmetry.

In *chapter 6*, our results will be further extended and embedded into the explicitly electromagnetic duality covariant framework of the embedding tensor formalism. Generalized Chern-Simons terms are manifestly present in this formalism, as well as the (non-relaxed) constraints on the shift transformations of the scalars. These constraints are now part of a more general consistency condition, called the representation constraint, which restricts the possible gaugings in the theory. Along the same lines as our treatment in chapter 5, we will study the consequences of relaxing this constraint, and we find that this is only possible if one also takes into account possible gauge anomalies. At the same time, we have obtained a physical interpretation for the representation constraint, which was still lacking: in its original form, the constraint simply reflects the absence of quantum gauge

[10] Generalized Chern-Simons terms were only considered in theories with extended supersymmetry in the past, but we show that they can also be present for $\mathcal{N} = 1$.

anomalies. Again these results are interesting for various applications. In particular they make contact with string compactifications in the presence of background fluxes, which may naturally lead to four-dimensional actions with tensor fields, gaugings in unusual duality frames and anomalous fermionic spectra. These aspects are naturally captured by the embedding tensor formalism with a modified representation constraint.

Finally, in *chapter 7* we return to the classical embedding tensor formalism in the absence of anomalies. As we noted before, this formalism has a complicated gauge structure involving vector fields ${A_\mu}^M(x)$, antisymmetric 2-forms ${B_{\mu\nu}}^{MN}(x)$ and several "types" of local transformations with parameters $\Lambda^M(x)$, ${\Xi_\mu}^{MN}(x)$ and ${\Phi_{\mu\nu}}^{MNP}(x)$. The principle goal of this chapter is to investigate the intricate gauge structure and to show that it generically involves open, soft and reducible algebra's. Open algebra's have commutators of two gauge transformations that only close up to terms that are proportional to the equations of motion of the fields:

$$[\delta_a, \delta_b]\phi^i = {f_{ab}}^c \delta_c \phi^i + \frac{\partial S}{\partial \phi^j} T^{ij}_{ab} \,. \tag{1.1}$$

We will see that for the embedding tensor formalism, there exist indeed non-trivial tensors $T^{ij}_{ab}$. Soft algebra's, on the other hand, arise when the structure constants are not really constant, but depend on the fields:

$$[\delta_a, \delta_b]\phi^i = {f_{ab}}^c(\phi) \delta_c \phi^i \,. \tag{1.2}$$

We will explicitly calculate the functions ${f_{ab}}^c(\phi)$ and we find a dependence on ${A_\mu}^M(x)$ and ${B_{\mu\nu}}^{MN}(x)$ in general. Finally, for reducible algebra's, the gauge transformations are not all independent:

$$(\delta_a \phi^i) Z^a + (\delta_b \phi^i) Z^b + \ldots = 0 \qquad \text{with } Z^a,\ Z^b, \ldots \text{ not all zero.} \tag{1.3}$$

The non-vanishing coefficients $Z^a$, $Z^b$, ... are called zero modes. Our system turns out to be higher stage reducible which means that not only are there non-trivial zero modes, but these zero modes are also not independent themselves. However, it is not clear if this hierarchy of higher stage zero modes breaks down after a finite number of steps, i.e., whether the algebra is finitely reducible. Besides a better understanding of the gauge structure of the embedding tensor formalism, the second goal of this chapter is to provide a more concise description for these complicated properties. For that purpose, we will present a formulation of the embedding tensor formalism in terms of the (classical) field-antifield formalism, which was originally constructed for the quantization of complicated gauge theories with exactly the properties (open, soft and reducible) that we discussed above. Ultimately, we find a very compact and unified formulation of the gauge structure relations in terms of one "master equation". Moreover, we have now all the tools available to initiate the quantization of generic gauged field theories.

**Note to the reader.** To conclude this introduction, we would like to emphasize that although a lot of the details in this thesis are very technical, the (nonspecialist) reader can still obtain a good idea about the main motivation, most important concepts, calculational techniques and underlying logic of our work, from scanning the introduction and conclusion to each chapter. Moreover, the entire content of chapter 2, as well as major parts of chapters 3 and 4 should be accessible to any theoretical physicist. For the expert reader, it might not be necessary to read the introductory chapters 2 and 3, whereas chapter 4 can be used to familiarize oneself with our notations and as a reminder about the details of gauged supergravities (including the embedding tensor formalism). The remaining chapters 5 – 7 contain an overview of our research results. In general, the technical parts and difficult concepts in this thesis will be approached in a gradual way, and at several instances, we have illustrated our results by means of simple examples. We hope these efforts contribute to a better understanding, and make the reader's journey through (some parts of) this manuscript a pleasant and instructive experience.

CHAPTER 2

# GAUGE THEORIES

In the introduction we have emphasized the role of gauge theories in our search for a better (and unified) description of nature's laws. In particular, we have argued that they give us a deeper insight into the universal structure of all elementary particle interactions. Gauge theories are also the starting point for most of our results in this thesis, and therefore we have reserved this first chapter to explain their main properties in a way that is accessible to readers with a limited knowledge of the subject. Unfortunately, we will not be able to convey the full scale of their physical applications, since that would require the development of a *quantum* gauge theory. Rather, we will elaborate on their formal structure using a classical field theory formulation, only briefly mentioning some quantum properties at the end.

The outline of our discussion is as follows. Section 2.1 introduces classical electromagnetism, which is the easiest gauge theory around. This section should be seen as an invitation for the reader to get acquainted with our notations and terminology. Once we have familiarized ourselves with the easiest case, we will continue to study the structure of more general gauge theories that describe e.g. the weak and strong nuclear forces. For that purpose we introduce in §2.2 a standard procedure called the "gauge principle", and use it to construct general Yang-Mills theories in §2.3. Finally, in preparation to some of our results in chapters 5 and 6, we present a brief introduction to certain quantum aspects of gauge theories. In particular, in §2.4 we discuss gauge anomalies, which are inconsistencies that become apparent at the quantum level. We conclude this chapter with a short summary and outlook.

## 2.1 Invitation: gauge invariance in classical electromagnetism

In 1864 Maxwell published his famous work on "A dynamical theory of the electromagnetic field" [13]. In this paper he presented a set of first-order differential equations which form the basis of all classical electromagnetic phenomena. They take the following form (in SI units):

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho_{\mathrm{em}}}{\epsilon_0} \qquad \text{(Coulomb's law)} \tag{2.1}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0 \qquad \text{(Faraday-Lenz laws)} \tag{2.2}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \qquad \text{(no magnetic charges)} \tag{2.3}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} - \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} = \mu_0 \boldsymbol{j}_{\mathrm{em}} \qquad \text{(modified Ampère's law)} \tag{2.4}$$

These equations provide an intricate relationship between the electric and magnetic fields, $\boldsymbol{E}(\boldsymbol{x}, t)$ and $\boldsymbol{B}(\boldsymbol{x}, t)$, and the total charge density $\rho_{\mathrm{em}}(\boldsymbol{x}, t)$ and total current density $\boldsymbol{j}_{\mathrm{em}}(\boldsymbol{x}, t)$. In particular, in the absence of charged matter, (2.1)-(2.4) reduce to the electromagnetic wave equations in vacuum.

In classical electrodynamics, one is usually interested in the behavior of the electric and magnetic fields in the background of a space-time-dependent distribution of charged particles. That means one has to solve the differential equations (2.1)-(2.4) for given $\rho_{\mathrm{em}}$, $\boldsymbol{j}_{\mathrm{em}}$ and appropriate boundary conditions. Our interest in Maxwell theory will be more formal, though. Since it is the simplest example of a gauge theory, it provides an easy introduction to useful concepts, notations and terminology. Hopefully, this gradual approach will then elucidate the discussion of more general gauge theories later on.

### The electric potential and gauge invariance

Let us take a slow start and recall that Maxwell's equations are invariant under global Poincaré (and more generally, conformal) transformations. This invariance can be made manifest by introducing the Lorentz covariant field strength and covariant current:[1]

$$F^{0i} \equiv E^i\,, \qquad F^{ij} \equiv \varepsilon^{ijk} B_k \quad \text{with} \quad \varepsilon^{123} = 1\,, \tag{2.5}$$

$$J^\mu_{\rm em} \equiv (\rho_{\rm em}, \boldsymbol{j}_{\rm em})\,. \tag{2.6}$$

Then one can check that Maxwell's equations are equivalent to the expressions

$$\begin{cases} \varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 & \text{(homogeneous equations (2.2) and (2.3)),} \\ \partial_\nu F^{\mu\nu} = J^\mu_{\rm em} & \text{(equations (2.1) and (2.4)).} \end{cases} \tag{2.7}$$

An important consequence of the covariant formulation is that in Minkowski space-time, the field strength can (locally) always be written as the derivative of some space-time dependent 1-form $A_\mu(x)$,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{2.8}$$

This follows from the homogeneous equation in (2.7) and Poincaré's lemma, see [14]. The field $A_\mu(x)$ is called the *electric potential* and its components are related to the electric and magnetic fields in the following way:

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}\,, \qquad \boldsymbol{E} = -\boldsymbol{\nabla}\Phi - \frac{\partial \boldsymbol{A}}{\partial t}\,, \qquad \text{with } \; A_\mu = (\Phi, \boldsymbol{A}). \tag{2.9}$$

However, the potentials $\Phi$ and $\boldsymbol{A}$ are *not unique* for given physical fields $\boldsymbol{E}$ and $\boldsymbol{B}$. Indeed, one may transform $\Phi$ and $\boldsymbol{A}$ while preserving $\boldsymbol{E}$ and $\boldsymbol{B}$. More precisely, one can check that for an arbitrary function $\theta(x)$, the change

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \theta(x) \tag{2.10}$$

leaves the field strength $F_{\mu\nu}$ –and hence $\boldsymbol{E}$ and $\boldsymbol{B}$– invariant. Such a transformation is called a *gauge transformation*, and Maxwell's theory is said to be *gauge invariant*. It is important to notice that the shift of the potential in (2.10) is a *local* transformation, in the sense that it depends on the space-time position via the parameter $\theta(x)$.

The reader might wonder what it means to have a theory that is gauge invariant. Clearly, gauge transformations reveal a redundancy in the physical content of the potential $A_\mu$, i.e., the fields $A_\mu$ that are related by gauge transformations describe the same physics. In order to get rid of this redundancy, one may impose a condition on the four components of $A_\mu$, known as the gauge fixing condition. This constraint eliminates the unphysical degrees of freedom: together

[1] From now on we will work in the more convenient Heaviside-Lorentz system for electrical quantities (such as charge, current,...), and we use Natural Units for mass, length and time. In particular, we will set $\epsilon_0 = 1$, $\mu_0 = 1$ and $c = 1$. More info about this choice of units can be found in appendix A.

with the equations of motion, it guarantees the existence of only two independent polarization vectors. These correspond to the two transversal polarization directions (or two helicity states) of electromagnetic waves.

Of course, one could also have avoided the gauge redundancy from the start, using the original formulation in terms of the physical fields **E** and **B**. However, there are a few reasons why the introduction of the gauge potential is ultimately unavoidable.

1. The construction of Lorentz covariant interactions of the electromagnetic field with charged particles requires the presence of $A_\mu$.
2. The components of $A_\mu$ are the dynamical degrees of freedom in a covariant Lagrangian description of electromagnetism.
3. The generalization to more complicated (Yang-Mills) gauge theories requires the vector fields from the start.
4. In a quantum theory of electromagnetism, the potential contains more information than the fields $\boldsymbol{E}$ and $\boldsymbol{B}$. It has some observable effects that were first confirmed by an experimental test of the Aharonov-Bohm effect [15,16].

In the next section, we will discuss the first two items. In particular, we will see that gauge invariance is a powerful tool to construct the interactions between radiation and matter.

### Gauge invariant matter couplings

First, we recall that the free Maxwell Lagrangian takes the form

$$\mathcal{L}_\gamma = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \,. \tag{2.11}$$

This is the only Lorentz scalar that is quadratic in the field strengths. The first Maxwell equation in vacuum (i.e. the second line of (2.7) with $J^\mu_{\mathrm{em}} = 0$) follows from an Euler-Lagrange variation of $\mathcal{L}_\gamma$ with respect to $A_\mu$. The homogeneous Maxwell equation (i.e. the first line of (2.7)) is an immediate consequence of the definition (2.8). Moreover, we note that $\mathcal{L}_\gamma$ is also gauge invariant.

Next, we would like to extend $\mathcal{L}_\gamma$ to a Lagrangian that describes the interaction of the electromagnetic field with charged matter (such as electrons). This requires the addition of two extra parts, one that describes the free matter fields, and one that characterizes the interaction with radiation:

$$\mathcal{L}_{\mathrm{em}} = \mathcal{L}_\gamma + \mathcal{L}_{\mathrm{matter}} + \mathcal{L}_{\mathrm{int}} \,. \tag{2.12}$$

The remainder of this section deals with the precise structure of these new parts. We will see that Lorentz and gauge invariance strongly constrain the expression for $\mathcal{L}_{\text{int}}$.

First, we notice that in order to reproduce the non-homogeneous Maxwell equation in (2.7), the interaction term has to be proportional to the potential $A_\mu$:

$$\mathcal{L}_{\text{int}} = J^\mu_{\text{em}} A_\mu \,. \tag{2.13}$$

Here, $J^\mu_{\text{em}} = J^\mu_{\text{em}}[\Phi]$ is some functional of the matter fields $\Phi^i(x)$ with $i$ an index that enumerates the different matter species.[2]

So, the introduction of the electromagnetic potential $A_\mu$ is crucial for the existence of the interaction term $\mathcal{L}_{\text{int}}$. However, we remark that the form of $\mathcal{L}_{\text{int}}$ is somewhat delicate, because it does not obviously transform as a scalar under Lorentz transformations. Indeed, although the current $J^\mu_{\text{em}}$ transforms as a four-vector, the potential $A_\mu$ does not. It is only a four-vector up to a gauge transformation, which should be clear from its definition (2.8) and the covariant transformation of the field strength:

$$A_\mu(x) \quad \to \quad A'_\mu(x') = \Lambda^\nu{}_\mu A_\nu(\Lambda x') + \partial'_\mu \theta(\Lambda x') \,. \tag{2.14}$$

In order to restore Lorentz invariance, we shall require that the part of the action for matter and its interaction with radiation (i.e. $\mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{int}}$) be invariant under general gauge transformations (2.10). This will clearly put some constraints on the form of the interaction. Indeed, the infinitesimal variation under (2.10) is given by

$$\delta(\mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{int}}) = J^\mu_{\text{em}} \partial_\mu \theta(x) = -\theta(x) \partial_\mu J^\mu_{\text{em}} + \text{total derivative} \,. \tag{2.15}$$

Therefore, the action is gauge invariant (and therefore also Lorentz invariant) provided that $J^\mu_{\text{em}}$ is a conserved current,

$$\partial_\mu J^\mu_{\text{em}}[\Phi] = 0 \,. \tag{2.16}$$

Now what type of matter theories supply us with a conserved four-vector to which we can couple $A_\mu$? An answer to this question is provided by Noether's theorem, which tells us that to each internal global symmetry of a Lorentz invariant theory, we can associate a conserved four-vector (at least when the matter fields satisfy the equations of motion) [17]. Therefore, we are looking for a Lagrangian $\mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{int}}$ that is invariant under some global transformation of the fields, say

$$\delta \Phi^i = \theta f^i[\Phi], \tag{2.17}$$

[2] In ordinary Maxwell theory, the matter fields are fermions (electrons, ...) and the current $J^\mu_{\text{em}}$ does not depend on the potential $A_\mu$. In more complicated theories (such as the coupling of electromagnetism to scalars), there might be an interaction term proportional to $A_\mu A^\mu$. We will come back to this issue in §2.1.

for a constant parameter $\theta$ and some function $f^i$ of the fields and their derivatives. Then Noether's theorem tells us that the associated current has the following form:

$$J^{\mu}_{\text{Noether}} = -\frac{\partial\left(\mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{int}}\right)}{\partial(\partial_\mu \Phi^i(x))} f^i[\Phi]\,, \tag{2.18}$$

and $\partial_\mu J^\mu_{\text{Noether}} = 0$ if the equations of motion for the $\Phi^i$ are satisfied. Then gauge invariance –and hence Lorentz invariance– can be restored once we make the following assumptions:

1. The electromagnetic current $J^\mu_{\text{em}}$ that couples to the potential $A_\mu$ is proportional to the Noether current. The proportionality constant can be absorbed into the scale of the charges in the theory, and therefore we will set both currents equal:
$$J^\mu_{\text{em}} = J^\mu_{\text{Noether}}\,. \tag{2.19}$$
If the equations of motion for the $\Phi^i$ are satisfied, the vanishing of $\partial_\mu J^\mu_{\text{Noether}}$ and the identification (2.19) are enough to show gauge invariance of the matter Lagrangian (see (2.15)).

2. If we want to obtain gauge invariance without using the equations of motion for $\Phi^i$, we need to promote the global symmetry transformations (2.17) of the matter fields to *local gauge transformations* with space-time dependent parameters $\theta(x)$,
$$\delta\Phi^i = \theta(x) f^i[\Phi]\,. \tag{2.20}$$
Under these transformations the matter Lagrangian transforms as
$$\delta\left(\mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{int}}\right) = -J^\mu_{\text{Noether}}\partial_\mu\theta(x)\,. \tag{2.21}$$
With the identification in (2.19), this variation precisely cancels the previous contribution (2.15) from the gauge transformation of the electromagnetic potentials.

Let us summarize our results so far. A Lorentz covariant coupling of the electromagnetic field to some matter fields is only possible if there exists a global symmetry (2.17) in the matter sector, and the coupling is proportional to the associated conserved current. Moreover, if we want to show gauge invariance without the use of the equations of motion of the matter fields, the global symmetry has to be promoted to a local transformation of these fields. In combination with the local transformation of the electromagnetic potential, it leads to a gauge invariant (and hence Lorentz invariant) Lagrangian.

### Examples

In order to further clarify the prominent role of Lorentz and gauge invariance in the construction of interactions with electromagnetic radiation, let us illustrate our construction by means of two examples. In the first example, we want to couple a complex Dirac spinor[3] $\Psi(x)$ to the electromagnetic field. This model describes classical electrodynamics, with the fermion $\Psi(x)$ a spin-1/2 particle such as the electron, and $A_\mu(x)$ a photon. The free fermion Lagrangian leads to the Dirac equation and is therefore given by

$$\mathcal{L}_\mathrm{D} = -\bar{\Psi}\gamma^\mu\partial_\mu\Psi\,. \tag{2.22}$$

This Lagrangian is invariant under global phase transformations:

$$\Psi(x) \to \Psi'(x) \equiv \mathrm{e}^{\mathrm{i}q\theta}\Psi(x)\,, \tag{2.23}$$

with $q$ some real constant which will later be identified with the charge of the fermions. Following the notation in (2.17), we find that $f = \mathrm{i}q\,\Psi(x)$. In order to determine the interaction with radiation, we need to solve (2.19) for the electromagnetic current. Using the definition (2.18), we find

$$J^\mu_\mathrm{em} = \left(\bar{\Psi}\gamma^\mu - \frac{\partial J^\nu_\mathrm{em}}{\partial(\partial_\mu\Psi)}A_\nu\right)\mathrm{i}q\Psi\,. \tag{2.24}$$

A solution to this equation can easily be found:

$$J^\mu_\mathrm{em} = \mathrm{i}q\bar{\Psi}\gamma^\mu\Psi\,. \tag{2.25}$$

Then it is a straightforward exercise to check that the Lagrangian

$$\begin{aligned}\mathcal{L}_\mathrm{D} + \mathcal{L}_\mathrm{int} &= -\bar{\Psi}\gamma^\mu\partial_\mu\Psi + \left(\mathrm{i}q\bar{\Psi}\gamma^\mu\Psi\right)A_\mu \\ &= -\bar{\Psi}\gamma^\mu\left(\partial_\mu - \mathrm{i}qA_\mu\right)\Psi \end{aligned}\tag{2.26}$$

is indeed invariant under the gauge transformations

$$\delta A_\mu(x) = \partial_\mu\theta(x)\,, \qquad \delta\Psi(x) = \mathrm{i}q\,\theta(x)\,\Psi(x)\,. \tag{2.27}$$

Therefore, we conclude that (2.26) is also Lorentz invariant.

The final result (2.26) turns out to have one undetermined parameter, $q$. Because the value of $q$ is not fixed by classical gauge invariance, different fermions $\Psi$ can have different values for $q$. Then what is the meaning of $q$? First we notice that it fixes the strength of the interaction between photons and fermions. If we set $q = 0$, the interaction vanishes. However, the true physical meaning only becomes clear

[3]More information about spinors and our conventions can be found in appendix A.

in a full quantum theory of electromagnetism. Then the operator $Q \equiv \int \mathrm{d}^3\vec{x}\, J^0_{\mathrm{em}}$ counts the number of particles minus the number of antiparticles, multiplied by $q$. Therefore, $Q$ can be interpreted as the charge operator, and $(-)q$ as the charge of the (anti)fermions. Moreover, the continuity equation $\partial_\mu J^\mu_{\mathrm{em}} = 0$ expresses the conservation of charge.

In the second example, we study a slightly more complicated situation where $J^\mu_{\mathrm{em}}$ in (2.13) depends on the potential $A_\mu$. In that case, (2.19) needs to be modified to

$$J^\mu_{\mathrm{em}} + \frac{\partial J^\nu_{\mathrm{em}}}{\partial A_\mu} A_\nu = J^\mu_{\mathrm{Noether}} \,. \tag{2.28}$$

The example we have in mind is the coupling of a complex scalar field to radiation. The free scalar Lagrangian is given by

$$\mathcal{L}_{\mathrm{KG}} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi^* \,, \tag{2.29}$$

where the subscript KG stands for "Klein-Gordon". This Lagrangian is invariant under global phase transformations of the fields:[4]

$$\delta\phi(x) = \mathrm{i}q\,\theta\,\phi(x)\,, \qquad \delta\phi^*(x) = -\mathrm{i}q\,\theta\,\phi^*(x)\,, \tag{2.30}$$

with $q$ some real constant which we can again identify with the charge of the scalar particles. Following (2.17), we propose $f = \mathrm{i}q\phi(x)\,,\ f^* = -\mathrm{i}q\phi^*(x)$. In order to determine the interaction with radiation, we need to solve (2.28) for the current $J^\mu_{\mathrm{em}}$. Using the definition (2.18), we find

$$\begin{aligned} J^\mu_{\mathrm{em}} + \frac{\partial J^\nu_{\mathrm{em}}}{\partial A_\mu} A_\nu \quad &= \quad -\frac{\partial J^\nu_{\mathrm{em}}}{\partial(\partial_\mu\phi)} A_\nu(\mathrm{i}q\phi) + \frac{1}{2}\partial^\mu\phi^*(\mathrm{i}q\phi) \\ &\qquad -\frac{\partial J^\nu_{\mathrm{em}}}{\partial(\partial_\mu\phi^*)} A_\nu(-\mathrm{i}q\phi^*) + \frac{1}{2}\partial^\mu\phi(-\mathrm{i}q\phi^*)\,. \end{aligned} \tag{2.31}$$

After some trial and error, the following solution to this equation can be found:

$$J^\mu_{\mathrm{em}} = \frac{1}{2}\mathrm{i}q\,\phi\partial^\mu\phi^* - \frac{1}{2}\mathrm{i}q\,\phi^*\partial^\mu\phi - \frac{1}{2}q^2 A^\mu\phi\phi^* \,. \tag{2.32}$$

Then it is an easy exercise to check that the Lagrangian

$$\begin{aligned} \mathcal{L}_{\mathrm{KG}} + \mathcal{L}_{\mathrm{int}} \quad &= \quad -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi^* + \left(\frac{1}{2}\mathrm{i}\phi\partial^\mu\phi^* - \frac{1}{2}\mathrm{i}\phi^*\partial^\mu\phi - \frac{1}{2}qA^\mu\phi\phi^*\right) qA_\mu \\ &= \quad -\frac{1}{2}\left(\partial_\mu - \mathrm{i}qA_\mu\right)\phi\left(\partial^\mu + \mathrm{i}qA^\mu\right)\phi^* \end{aligned} \tag{2.33}$$

[4]Remark that *real* scalar fields cannot be coupled to the electromagnetic field and are therefore always uncharged. We can now understand why; the Lagrangian $\mathcal{L}_{\mathrm{KG}} = -\frac{1}{2}(\partial_\mu\phi)^2$ has no global continuous symmetries as in (2.23), and therefore does not allow a Lorentz covariant coupling of $\phi$ to $A_\mu$.

is invariant under the gauge transformations

$$\delta A_\mu(x) = \partial_\mu \theta(x)\,,$$
$$\delta\phi(x) = \mathrm{i}q\,\theta(x)\,\phi(x)\,, \qquad \delta\phi^*(x) = -\mathrm{i}q\,\theta(x)\,\phi^*(x)\,. \tag{2.34}$$

One can imagine that for more complicated matter Lagrangians, it is not an easy exercise to solve equation (2.28). As such, in more general theories it is not straightforward to determine the precise coupling to electromagnetic radiation following this method. However, there does exist a different procedure –called the gauging procedure– that allows for the construction of these interactions in a way that is more transparent and easier to accomplish. Although less intuitive, the gauging procedure is much more powerful and has led to numerous successes in the recent history of high energy physics. We will outline its basic properties in the next section.

## 2.2 The gauge principle

In the previous section, we started with the existence of massless spin-one particles (photons) and determined their coupling to other matter fields (in particular Dirac fermions and complex scalars). An important guideline to construct the interaction term was the invariance of the theory under Lorentz and gauge transformations.

In this section, we will reverse the argument and start from a *matter theory* with a set of global symmetries. For example, consider the Lagrangian (2.26) for free fermions again,

$$\mathcal{L}_{\mathrm{D}} = -\bar{\Psi}\gamma^\mu\partial^\mu\Psi\,, \tag{2.35}$$

and recall its invariance under the global field transformations

$$\Psi(x) \quad\rightarrow\quad \Psi'(x) = \mathrm{e}^{\mathrm{i}q\,\theta}\Psi(x)\,, \tag{2.36}$$

with $\theta$ a constant parameter. These are $U(1)$ phase transformations.

At this point one might wonder what it means to have a physical theory that is left invariant under global phase transformations. If one fixes the phase at one space-time point, one is then not free to make an arbitrary choice at any other space-time point. This sounds like a limitation that is not consistent with the localized field concept. In order to resolve this issue, C. N. Yang and R. Mills [18] revived an old idea of H. Weyl [19] and proposed to extend the global phase invariance to a local phase invariance, i.e., to look for a theory that is invariant under transformations with a space-time dependent parameter $\theta(x)$. Of course, the original Lagrangian $\mathcal{L}_{\mathrm{D}}$ is *not* invariant under local phase transformations and therefore, one has to introduce new physics. Remarkably, the new physical ingredients are precisely the matter-radiation interactions that we discussed in the previous section. This procedure is therefore called the *gauging procedure* or *principle of minimal coupling*. A detailed derivation of this result can be found in numerous textbooks (see for example [20–23]) and therefore, we will only briefly present the basic steps.

We start from the variation of $\mathcal{L}_\mathrm{D}$ under *local* phase transformations:

$$\delta\mathcal{L}_\mathrm{D} = -\mathrm{i}q\, \bar{\Psi}(x)\gamma^\mu\partial_\mu\theta(x)\,. \tag{2.37}$$

The non-trivial contribution comes from the derivative in the Lagrangian. In order to restore invariance, we therefore have to change the ordinary space-time derivative to a *covariant derivative*. The latter is defined by its transformation under local phase changes:

$$\delta\left(D_\mu\Psi\right) = \mathrm{i}q\theta(x)D_\mu\Psi\,, \tag{2.38}$$

i.e., the covariant derivative transforms in the same way as the fields. Once we have replaced the ordinary derivative in (2.35) by a covariant one, the new Lagrangian is invariant under local phase transformations.

So, the only remaining task is to construct an appropriate covariant derivative. Usually, this is done via geometrical arguments, similar to the construction of covariant derivatives in general relativity (see e.g. [20]). It requires the introduction of a connection and its non-covariant transformation. Loosely speaking, the connection describes the relation between phase values of the fields at nearby points. In this case, we find

$$D_\mu\Psi(x) \equiv \left(\partial_\mu - \mathrm{i}qA_\mu(x)\right)\Psi(x)\,, \qquad \delta A_\mu(x) = \partial_\mu\theta(x)\,. \tag{2.39}$$

The one-form connection $A_\mu(x)$ is a real space-time dependent field, which can be identified with the electromagnetic potential. Its transformation in (2.39) is identical to the gauge transformation in (2.10) and guarantees that (2.38) is indeed satisfied.

Given the expression for the covariant derivatives, the new invariant Lagrangian is

$$\mathcal{L} \equiv \mathcal{L}_\mathrm{D}\big|_{\partial_\mu\to D_\mu} = -\bar{\Psi}(x)\gamma^\mu\left(\partial_\mu - \mathrm{i}qA_\mu(x)\right)\Psi(x)\,. \tag{2.40}$$

Not surprisingly, $\mathcal{L}$ has exactly the same form as the Lagrangian in (2.26). The new term that was added through the covariant derivative, corresponds to $\mathcal{L}_\mathrm{int}$ in the previous section, and describes the interaction of the fermions with radiation. Finally, we still have to add the Lagrangian (2.11) for the free electromagnetic field. A nice way to define the field strength $F_{\mu\nu}$ is via

$$[D_\mu, D_\nu]\Psi = -\mathrm{i}qF_{\mu\nu}\Psi\,. \tag{2.41}$$

An explicit calculation of the left hand side reveals that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, exactly as expected.

To convince the reader that the gauge procedure is indeed a powerful method, let us apply it once again to the second example from the previous section. We take the free scalar Lagrangian which is invariant under global $U(1)$ transformations (2.17), and replace the ordinary derivatives by covariant ones:

$$\mathcal{L} = -\frac{1}{2} D_\mu \phi D^\mu \phi^* \,; \qquad D_\mu \phi = (\partial_\mu - \mathrm{i} q A_\mu)\, \phi \,. \tag{2.42}$$

Clearly this Lagrangian is invariant under the gauge transformations (2.34) and we immediately recover the interaction terms $\mathcal{L}_{\text{int}}$ from the previous section.

So far, we have illustrated the basic philosophy behind the gauging procedure by means of two simple examples. In both cases, the global symmetry group of the original matter theory was an Abelian group, leading to a Maxwell-type interaction with one $U(1)$ vector field $A_\mu$. However, we have not learned anything new; the gauge principle approach was only an interesting way of looking at an interaction whose form was originally determined in other (empirical) ways. It was only in 1954 that Yang and Mills took the gauge principle and used it as a method to construct *new* physical theories [18]. They extended the procedure to more complicated (non-Abelian) global symmetry groups and their gauging. Ultimately, their work lead to a very successful description of all fundamental forces in nature (except gravity). These theories were called Yang-Mills gauge theories and we will discuss their classical properties in the next section.

## 2.3 Yang-Mills theories

The paradigmatic example of a Yang-Mills gauge theory is the Standard Model of elementary particle interactions. An integral part of the Standard Model is Quantum Chromodynamics, which describes the strong interaction between quarks and gluons, and gluons among each other. The form of these interactions is fixed by the gauge principle, much in the same way as it fixes the interaction between electrons and photons. The main difference though, is the more complicated type of gauge group. QCD has a non-Abelian $SU(3)$ gauge symmetry, which leads to a richer structure, but also to several complications in the construction.

In this section, we will review the construction of generic Yang-Mills theories, thereby extending the gauge principle to general non-Abelian gauge groups. The focus of our discussion will be on the basic formulas, and we emphasize the aspects that are most relevant for further developments in later chapters of this thesis. Again we do not enter into the underlying geometric ideas and we postpone a discussion about possible quantum effects to the next section.

### Matter content and global symmetries

We start from a generic set of Dirac spinors $\Psi^A$ and complex scalars $\phi^i$. The fields transform in an irreducible representation of some continuous group $G$, as indicated by their upper indices[5] $A$ and $i$. For simplicity, we assume that $G$ is a compact simple Lie group[6] of dimension $\dim_G$, and the (conjugate) fields transform in certain matrix representations:

$$\delta\Psi^A = -\theta^\Sigma (t_\Sigma)^A{}_B \Psi^B\,, \qquad \delta\bar{\Psi}_A = \bar{\Psi}_B \theta^\Sigma (t_\Sigma)^B{}_A\,, \tag{2.43}$$

$$\delta\phi^i = -\theta^\Sigma (t_\Sigma)^i{}_j \phi^j\,, \qquad \delta\phi^*_i = \phi^*_j \theta^\Sigma (t_\Sigma)^j{}_i\,. \tag{2.44}$$

Here, $\Sigma = 1, \ldots, \dim_G$ labels the generators of $G$. The anti-hermitian matrices $(t_\Sigma)^A{}_B = -(t^*_\Sigma)^B{}_A$ and $(t_\Sigma)^i{}_j = -(t^*_\Sigma)^j{}_i$, which are the generators in the representations of the fields, satisfy the commutation relations

$$[t_\Lambda, t_\Sigma] = f_{\Lambda\Sigma}{}^\Xi t_\Xi\,, \tag{2.45}$$

with $f_{\Lambda\Sigma}{}^\Xi = -f_{\Sigma\Lambda}{}^\Xi$ the structure constants. Consistency requires that the latter obey the Jacobi identity

$$f_{\Lambda\Omega}{}^\Gamma f_{\Sigma\Xi}{}^\Omega + f_{\Sigma\Omega}{}^\Gamma f_{\Xi\Lambda}{}^\Omega + f_{\Xi\Omega}{}^\Gamma f_{\Lambda\Sigma}{}^\Omega = 0\,. \tag{2.46}$$

Finally, the $\theta^\Sigma$ in (2.43) and (2.44) are real constant parameters, one for each generator.

Let us now presume the existence of a Lagrangian $\mathcal{L}_{\text{matter}}$ that describes the dynamics of the fields and their conjugates. Furthermore, we will assume that $\mathcal{L}_{\text{matter}}$ is invariant under the global transformations (2.43) and (2.44). A common example of such an invariant Lagrangian is given by the free fields

$$\mathcal{L}_{\text{matter}} = -\bar{\Psi}_A \gamma^\mu \partial_\mu \Psi^A - \frac{1}{2}\partial_\mu \phi^i \partial^\mu \phi^*_i\,. \tag{2.47}$$

This is the analogue of the Dirac and Klein Gordon Lagrangians in (2.22) and (2.29) respectively. The field transformations in (2.43)-(2.44) are an extension of the global phase transformations, $\delta\Psi = \mathrm{i}q\theta\Psi$ and $\delta\phi = \mathrm{i}q\theta\phi$.

Given the Lagrangian $\mathcal{L}_{\text{matter}}$ that is invariant under $G$, one can now use the gauge principle to introduce the interaction between scalars, fermions and certain spin-1 vector fields.

[5]The representation indices $A$ and $i$ take values from 1 to the dimension of the representation. We emphasize that $A$ should not be confused with a spinor index which we have omitted here. We also note that the conjugated fields will be labeled by lower indices, i.e., $\bar{\Psi}_A$ and $\phi^*_i$.

[6]Simple Lie groups are defined to be non-Abelian, which is what we are interested in. The motivation for studying compact Lie groups will be discussed below.

### The gauging procedure

The first step requires the promotion of the global symmetry group $G$ to a set of local transformations. These form a group which we denote by $G_{\rm local}$. In general, $G_{\rm local}$ is only a subset of the full global symmetry group, i.e. $G_{\rm local} \subseteq G$. However, to simplify things, we focus on the case where $G_{\rm local} = G$. The fermions and scalars take the following infinitesimal transformations under $G_{\rm local}$ (similar expressions apply to the conjugate fields):

$$\delta\Psi^A = -\theta^\Sigma(x)(t_\Sigma)^A{}_B\Psi^B\,, \qquad \delta\phi^i = -\theta^\Sigma(x)(t_\Sigma)^i{}_j\phi^j\,. \tag{2.48}$$

This time, the $\dim_G$ parameters $\theta^\Sigma(x)$ are space-time dependent. Therefore, the Lagrangian $\mathcal{L}_{\rm matter}$ is not invariant under these transformations, but picks up extra contributions that are proportional to the space-time derivative of $\theta^A(x)$.

In order to obtain a gauge invariant theory, we have learned from §2.2 that it is sufficient to replace every ordinary derivative $\partial_\mu$ by a covariant derivative $D_\mu$. The reason is that covariant derivatives do not transform with the derivative of a parameter under $G_{\rm local}$. Symbolically, they take the following form:[7]

$$D_\mu \equiv \partial_\mu - gA_\mu{}^\Sigma\delta_\Sigma\,, \tag{2.49}$$

with $\delta_\Sigma$ the generators of $G_{\rm local}$ and $A_\mu{}^\Sigma$ the corresponding 1-form connections. When acting on some given field which transforms according to a particular representation of the group $G$, one replaces the $\delta_\Sigma$ by the relevant representation matrices. Thus acting on spinors and scalars, $D_\mu$ is represented by

$$D_\mu\Psi^A = \partial_\mu\Psi^A + gA_\mu{}^\Sigma(t_\Sigma)^A{}_B\Psi^B\,, \tag{2.50}$$

$$D_\mu\phi^i = \partial_\mu\phi^i + gA_\mu{}^\Sigma(t_\Sigma)^i{}_j\phi^j\,. \tag{2.51}$$

The form of these non-Abelian covariant derivatives is a generalization of (2.39) and (2.42).

We stress that for each local symmetry operator $\delta_\Sigma$, $\Sigma = 1,\ldots,\dim_G$, we have introduced a corresponding connection $A_\mu{}^\Sigma$. Each of these spin-1 gauge fields (or gauge bosons) describes a different force mediating particle. Each of them also has its own gauge transformation,

$$\delta A_\mu{}^\Sigma = \partial_\mu\theta^\Sigma(x) - g\theta^\Lambda(x)A_\mu{}^\Omega f_{\Lambda\Omega}{}^\Sigma\,. \tag{2.52}$$

The first term contains a derivative on the gauge parameters, identical to the transformation of the $U(1)$ gauge field, see (2.10). The second term reflects the fact that $A_\mu{}^\Sigma$ carries an adjoint index, for which $(t_\Lambda)^\Sigma{}_\Omega = f_{\Lambda\Omega}{}^\Sigma$. The

[7] The parameter $g$ is the coupling constant of the theory, similar to the charge in electromagnetism.

total transformation $\delta A_\mu{}^\Sigma$ guarantees that (2.50) and (2.51) indeed transform as covariant derivatives.

If we apply the change $\partial_\mu \to D_\mu$ to our example in (2.47), we can explicitly check that

$$\begin{aligned}
\mathcal{L}_{\text{int}} &= -\bar{\Psi}_A \gamma^\mu D_\mu \Psi^A - \frac{1}{2} D_\mu \phi^i D^\mu \phi_i^* \\
&= \mathcal{L}_{\text{matter}} + \left[ -\bar{\Psi}_A \gamma^\mu (t_\Sigma \Psi)^A - \frac{1}{2} (t_\Sigma \phi)^i \, \partial^\mu \phi_i^* + \frac{1}{2} \partial^\mu \phi^i \, (\phi^* t_\Sigma)_i \right] g A_\mu{}^\Sigma \\
&\quad + \frac{1}{2} g^2 A_\mu{}^\Sigma A^{\mu\,\Omega} \, (t_\Sigma \phi)^i \, (\phi^* t_\Omega)_i
\end{aligned} \tag{2.53}$$

is indeed invariant under the gauge transformations (2.48) and (2.52). The extra interaction terms make sure that all derivatives on the gauge parameters $\theta^\Sigma(x)$ vanish.

Finally, we still have to add a term that describes the "free" gauge bosons $A_\mu{}^\Sigma$. In electromagnetism, the Lagrangian for free photons is proportional to a quadratic combination of two field strengths, see (2.11). The same applies to non-Abelian theories, where the Lagrangian must contain a term that is quadratic in $F_{\mu\nu}{}^\Sigma \equiv \partial_\mu A_\nu{}^\Sigma - \partial_\nu A_\mu{}^\Sigma$. However, for non-Abelian theories the ordinary field strengths $F_{\mu\nu}{}^\Sigma$ are not covariant anymore, which means we cannot use them to construct a gauge invariant combination. In order to solve this issue, we will construct non-Abelian covariant field strengths via

$$[D_\mu, D_\nu] \Psi^A = -g \mathcal{F}_{\mu\nu}{}^\Sigma (\delta_\Sigma \Psi)^A \,, \tag{2.54}$$

similar to (2.41). An explicit calculations shows that

$$\mathcal{F}_{\mu\nu}{}^\Sigma = \partial_\mu A_\nu{}^\Sigma - \partial_\nu A_\mu{}^\Sigma + g f_{\Lambda\Omega}{}^\Sigma A_\mu{}^\Lambda A_\nu{}^\Omega \,, \tag{2.55}$$

which is indeed the covariant generalization of $F_{\mu\nu}{}^\Sigma$. Given this expression, we can now construct the most general Lagrangian that is quadratic in the new field strengths $\mathcal{F}_{\mu\nu}{}^\Sigma$, and that is Lorentz invariant:

$$\mathcal{L}_A = -\frac{1}{4} f_{\Lambda\Sigma} \mathcal{F}_{\mu\nu}{}^\Lambda \mathcal{F}^{\mu\nu\,\Sigma} \,, \tag{2.56}$$

for some constant, real and symmetric matrix $f_{\Lambda\Sigma}$.[8]

If we combine this result with the expression for $\mathcal{L}_{\text{int}}$, where all ordinary space-time derivatives have been replace by covariant derivatives, we obtain the full

[8] In the context of supergravities, these gauge kinetic functions will not be constant in general, but they depend on the other fields in the theory, such as scalars.

Yang-Mills Lagrangian that is invariant under the local transformations (2.48) and (2.52):

$$\mathcal{L}_{\rm YM} = \mathcal{L}_A + \mathcal{L}_{\rm int}\,. \tag{2.57}$$

Given this expression for the total Lagrangian, we can investigate some important properties of Yang-Mills theories that will be relevant for the remainder of our discussion. In particular, we will examine the properties of the matrix $f_{\Lambda\Sigma}$, the addition of a "$\theta$-term" to the Lagrangian, and the conserved currents of the theory.

### Properties of Yang-Mills theories

First, we review the properties of the constant matrix $f_{\Lambda\Sigma}$. The kinetic Lagrangian $\mathcal{L}_A$ is only gauge invariant if $f_{\Lambda\Sigma}$ satisfies the condition

$$f_{\Lambda(\Sigma}\, f_{\Omega)\Xi}{}^{\Lambda} = 0\,. \tag{2.58}$$

Moreover, $f_{\Lambda\Sigma}$ has to be positive definite. These two constraints lead to some important consequences for the allowed gauge groups $G_{\rm local}$. One can show that $G_{\rm local}$ has to be a direct product of compact simple groups and $U(1)$ factors [21]. This clarifies our choice at the beginning of §2.3 to confine the discussion to compact simple groups. From a model-building perspective, this is an interesting restriction, since all compact simple groups have been classified and a great deal is known about their representations. Moreover, one can show that under the conditions (2.58) and positivity, the matrix $f_{\Lambda\Sigma}$ takes a special form for a proper normalization of the generators $t_\Lambda$. If we split the vector indices $\Lambda, \Sigma, \ldots$ into pairs $m\lambda, n\sigma, \ldots$ where $m$ and $n$ label the simple and $U(1)$ subalgebras, and $\lambda$ and $\sigma$ label the individual generators of these subalgebras, then one can show that

$$f_{\Lambda\Sigma} = f_{m\lambda,n\sigma} = g_{(m)}^{-2}\,\mathbb{1}_{mn}\mathbb{1}_{\lambda\sigma}\,, \tag{2.59}$$

with real $g_{(m)}$. The factors $1/g_{(m)}^2$ can be eliminated via a rescaling of the vector fields,

$$A_\mu{}^{ma} \quad\rightarrow\quad g_{(m)}^{-1} A_\mu{}^{ma}\,. \tag{2.60}$$

In order to preserve the relations (2.49) and (2.55), one must also redefine the generators $\delta_\Lambda$ and the structure constants $f_{\Lambda\Sigma}{}^{\Omega}$:

$$\delta_{ma} \;\rightarrow\; g_{(m)}\delta_{ma}\,, \qquad\qquad f_{ab}^{(m)c} \;\rightarrow\; g_{(m)} f_{ab}^{(m)c}\,. \tag{2.61}$$

In the end, the most general kinetic Lagrangian for the gauge bosons has the form

$$\mathcal{L}_A = -\frac{1}{4}\mathcal{F}_{\mu\nu}{}^{\Lambda}\mathcal{F}^{\mu\nu\Lambda}\,. \tag{2.62}$$

We should remark, though, that (2.62) is not the only combination that is quadratic in the field strengths and that fulfils the requirements of gauge- and

Lorentz invariance. Indeed, we can write down the so-called "$\theta$-term"

$$\mathcal{L}_\theta = \frac{1}{8}\theta_{\Lambda\Sigma}\,\varepsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}{}^{\Lambda}\mathcal{F}_{\nu\rho}{}^{\Sigma}\,, \tag{2.63}$$

with a different constant matrix that is conventionally denoted by $\theta_{\Lambda\Sigma}$. Since $\mathcal{L}_\theta$ has the desired properties, there is nothing that prevents us from adding it to the Lagrangian $\mathcal{L}_{\mathrm{YM}}$. On the other hand, for a constant matrix $\theta_{\Lambda\Sigma}$ this term is actually a total derivative, and therefore it does not effect the equations of motion or the perturbative quantum description of the theory.[9] The reason why we have mentioned the possibility of a $\theta$-term at all, is because it is a natural constituent of the supersymmetric theories we will encounter in chapter 3. In general, the matrix $\theta_{\Lambda\Sigma}$ will then be a function of the scalar fields, and $\mathcal{L}_\theta$ does not vanish anymore.

To finish this section, let us derive the equations of motion for the gauge bosons from $\mathcal{L}_{\mathrm{YM}}$,

$$D_\nu\left(f_{\Sigma\Lambda}\mathcal{F}^{\mu\nu\Lambda}\right) = J^\mu{}_\Sigma \qquad \text{with} \quad J^\mu{}_\Sigma = \frac{\partial\mathcal{L}_{\mathrm{int}}}{\partial A_\mu{}^\Sigma}\,. \tag{2.64}$$

This result looks very similar to the non-homogeneous Maxwell equation in (2.7). There are however a few fundamental differences.

- In contrast to electromagnetism, the matter currents $J^\mu{}_\Sigma$ are not conserved in the usual sense, i.e. $\partial_\mu J^\mu{}_\Sigma \neq 0$. However, it can be shown from (2.64) that

  $$D_\mu J^\mu{}_\Sigma = 0\,. \tag{2.65}$$

  This might seem strange because we after all, our theory is invariant under global transformations of $G$. Therefore, we expect from Noether's theorem that there exist $\dim_G$ conserved currents. This is indeed the case, but in contrast to electromagnetism, these currents are not simply given in terms of the matter fields anymore. Rather, they turn out to be

  $$\tilde{J}^\mu{}_\Sigma \equiv J^\mu{}_\Sigma + gA_\nu{}^\Xi f_{\Xi\Sigma}{}^\Omega f_{\Omega\Lambda}\mathcal{F}^{\mu\nu\,\Lambda}\,. \tag{2.66}$$

  Interestingly, the Noether currents now contain a term that is proportional to the covariant field strength $\mathcal{F}^{\mu\nu\,\Lambda}$. Ultimately, this is due to the $A_\mu{}^\Sigma$ being charged under the gauge group, in contrast to the photons which are electrically neutral.

- In electromagnetism, the conserved current $J^\mu_{\mathrm{em}}$ is gauge invariant. This can be checked explicitly for the examples in (2.25) and (2.32). In a general Yang-Mills theory, though, neither $J^\mu{}_\Sigma$ nor $\tilde{J}^\mu{}_\Sigma$ are gauge invariant. Both currents carry an adjoint group index and therefore transform accordingly. This is of course consistent with the equations of motion (2.64).

[9] It does have non-perturbative quantum effects though, but for more details about these, we refer to the literature.

- The non-homogeneous equations of motion (2.64) need to be supplemented by the Bianchi identity
$$\varepsilon^{\mu\nu\rho\sigma} D_\nu \mathcal{F}_{\rho\sigma}{}^{\Sigma} = 0\,. \qquad (2.67)$$
This is the non-Abelian counterpart of the homogeneous Maxwell equation in (2.7). It is again a gauge covariant expression.

This finishes our discussion about the relevant properties of classical Yang-Mills theories. In the next section, we will put these results into another perspective, thereby abandoning the classical setup and studying the *quantum effects* on gauge theories instead. The reason for our interest in these effects is mainly motivated by results that will be presented in later chapters. In particular, in chapters 5 and 6 we will see that certain supergravity theories (and more generally, string theories) require a nice interplay between both classical ánd quantum aspects.

## 2.4 Quantum anomalies

In previous parts of this text we have stressed the importance of both global and local symmetries in classical field theory. The global symmetries constrain the system, but do not uniquely determine it. On the other hand, gauge symmetries fix the interactions between matter and vector fields, but they also correspond to unphysical redundancies of the theory. In order to fully appreciate the physical relevance of these results, one needs to go beyond the classical description, and develop a corresponding *quantum field theory*. The question we try to answer is what happens to the global and local symmetries in such a quantum theory.

As we will see in §2.4, there is the possible effect of *quantum anomalies*, i.e., classical symmetries that disappear at the quantum level. For global symmetries, such anomalies do not pose any problems. Rather, they signal the presence of new physics, since the system is less constrained.[10] In the remainder of this text, we will therefore not be bothered by global anomalies (such as the chiral Adler-Bell-Jackiw anomaly) and their effects. For more information, we refer to excellent introductory texts such as [20, 21, 24].

Instead, we will concentrate on anomalous *gauge* symmetries, which are more problematic. Since gauge symmetries are needed to decouple the unphysical states of the theory, a violation of these symmetries renders the theory inconsistent. Therefore, gauge anomalies must be excluded from physical theories. In the Standard Model, for example, anomaly freedom is automatically achieved due to the particular field content and the associated charges. In more general theories such as supergravity or string theories, the absence of gauge anomalies is not

[10] One observable effect is the presence of Goldstone bosons, associated to the broken symmetry.

immediate but it requires an extra mechanism that involves the use of classical "counterterms" to cancel the anomaly. As it turns out, this mechanism can only be successfully applied to a limited number of theories and in particular, it led to the first "superstring revolution" [25] from which only five consistent superstring theories emerged. In other words, the requirement of anomaly freedom imposes strong constraints on the allowed physical theories. It is therefore important to study these constraints and improve our understanding about them. In this thesis we will be interested in the details of anomaly cancellation in 4-dimensional minimal supergravity in chapter 5, and its extension to theories with a generalized gauging in chapter 6.

In order to pave the way for these discussions, we will now review the origin and the general structure of gauge anomalies. In §2.4 we restrict to Maxwell and Yang-Mills type gauge theories using canonical quantization methods. In §2.4 we discuss the emergence of anomalies from a path integral approach, which will allow us to extend the results more easily to complicated theories such as supergravity.

### A first encounter with anomalies

Given the expressions for the $n$-point correlation functions in an interacting quantum field theory,[11]

$$G^{(n)}(x_1, \dots, x_n) \equiv \langle \hat{\Phi}(x_1) \dots \hat{\Phi}(x_n) \rangle \,, \tag{2.68}$$

one is able to calculate the vacuum expectation value of any field-dependent operator. In particular, one can study the quantum properties of an operator $\hat{J}^\mu$, which corresponds to a classical current such as $J^\mu_{\text{em}}$ in electromagnetism or $J^\mu{}_\Sigma$ in general Yang-Mills theories.

Recall from the previous section that these classical currents satisfy the continuity equations

$$\partial_\mu J^\mu_{\text{em}} = 0 \qquad \text{for Abelian gauge theories} \,, \tag{2.69}$$

$$D_\mu J^\mu{}_\Sigma = 0 \qquad \text{for Yang-Mills theories} \,, \tag{2.70}$$

which reflect the invariance of the theory under the corresponding gauge symmetries. A natural question to ask is whether these continuity equations also hold as quantum relations. In order to address this question more carefully, we will introduce the notation[12]

$$D_\mu \langle \hat{J}^\mu(x) \rangle = -\mathcal{A}(x) \,. \tag{2.71}$$

[11] The Heisenberg field operators will be denoted by $\hat{\Phi}(x)$. We also omit the vacuum and time ordering symbols, e.g. $\langle \hat{\Phi}(x_1)\hat{\Phi}(x_2) \rangle = \langle \Omega | T \hat{\Phi}(x_1)\hat{\Phi}(x_2) | \Omega \rangle$.

[12] The covariant derivative reduces to an ordinary derivative if $\hat{J}^\mu$ is an Abelian current.

The question above then boils down to studying the properties of the function $\mathcal{A}(x)$.

- If $\mathcal{A}(x) = 0$, the classical symmetry is also conserved at the quantum level.
- If $\mathcal{A}(x) \neq 0$, the theory is said to have an *anomaly* and the corresponding classical symmetry is no longer a symmetry of the quantum theory.

The calculation of $\mathcal{A}(x)$ in perturbation theory requires the evaluation of correlation functions (2.68), and the introduction of an appropriate regulator to obtain a finite result. If there exists such a regulator that is gauge invariant, the result for $\mathcal{A}(x)$ vanishes and there is no anomaly. On the other hand, if there is no gauge invariant regulator available, the result is non-zero and the theory is anomalous. This computation has been done for the easiest cases, i.e., for Abelian and Yang-Mills theories. Let us summarize the main results here.

The breakdown of gauge symmetries only occurs in theories with a coupling of chiral fermions to gauge fields.[13] Suppose we have a collection of left handed fermions $\psi_A^{(L)}$ $(A = 1, \ldots, n_L)$ that transform in the representation $\mathcal{R}_A^{(L)}$ of the gauge group $G_{\text{gauge}}$, and a set of right handed fermions $\psi_B^{(R)}$ $(B = 1, \ldots, n_R)$ that transform in the representation $\mathcal{R}_B^{(R)}$. Then one can show that the gauge currents

$$J^\mu{}_\Sigma = \bar{\psi}_A^{(L)} \gamma^\mu (t_\Sigma^{(L)}\, \psi^{(L)})^A + \bar{\psi}_B^{(R)} \gamma^\mu (t_\Sigma^{(R)}\, \psi^{(R)})^B \tag{2.72}$$

are generally not conserved in the quantum theory. One finds the following expression for the anomaly $\mathcal{A}_\Sigma(x)$:

$$-D_\mu \langle \hat{J}^\mu{}_\Sigma \rangle = \mathcal{A}_\Sigma(x) = -\frac{\hbar}{8} d_{\Sigma\Lambda\Omega} \varepsilon^{\mu\nu\rho\sigma} \left( F_{\mu\nu}{}^\Lambda F_{\rho\sigma}{}^\Omega + \mathcal{O}(A^3) \right) . \tag{2.73}$$

Let us analyze the different parts of this expression.

- The anomaly is a 1-loop quantum effect, which is apparent from its linear dependence on Planck's constant $\hbar$.
- The prefactor $d_{\Sigma\Lambda\Omega}$ is a constant, symmetric 3-tensor which is fixed by the representation content of the chiral fermions in the theory:

$$d_{\Sigma\Lambda\Omega} \equiv -\frac{\mathrm{i}}{12\pi^2} \left( \sum_{A=1}^{n_R} \mathrm{Tr}_{\mathcal{R}_A^{(R)}} \left( t_\Lambda \{ t_\Sigma, t_\Omega \} \right) - \sum_{B=1}^{n_L} \mathrm{Tr}_{\mathcal{R}_B^{(L)}} \left( t_\Lambda \{ t_\Sigma, t_\Omega \} \right) \right) . \tag{2.74}$$

[13] The theories from the previous section with Lagrangian $\mathcal{L}_{\text{YM}}$ do not fall into this class, since Dirac fermions are non-chiral. Therefore, these theories are automatically gauge anomaly-free.

The first summation runs over all left-handed fermions. For each of these fermions, a term is added that is given by the symmetric trace of three (anti-hermitian) group generators in the appropriate representation. The second summation runs over the right-handed fermions and is subtracted from the contribution of the left-handed fermions.

- The factor $d_{\Sigma\Lambda\Omega}$ is multiplied by the quadratic combination of 2-forms $F_{\mu\nu}{}^{\Sigma}$ and other terms that contain three or more gauge fields. The 2-forms are just the linear part of the full gauge covariant field strengths $\mathcal{F}_{\mu\nu}{}^{\Sigma}$. The $\mathcal{O}(A^3)$ contributions were not written down explicitly, but will be discussed in §5.4 when we need them.

Given the generic expression for the anomaly $\mathcal{A}_{\Sigma}$, we can now distinguish between anomaly-free theories for which $d_{\Sigma\Lambda\Omega} = 0$, and anomalous gauge theories for which $d_{\Sigma\Lambda\Omega} \neq 0$. Remember from the introduction that anomalous gauge transformations are unacceptable since they lead to an inconsistent quantum theory. Therefore, the only valuable quantum theories are those for which $d_{\Sigma\Lambda\Omega} = 0$. Clearly, this condition is only satisfied for some gauge groups and/or a particular fermion content.

First, the impact of the gauge group can be summarized as follows: if $G_{\text{gauge}}$ has only real or pseudoreal representations, then all traces in (2.74) vanish and $\mathcal{A}_{\Sigma}$ is automatically zero, irrespective of the fermion content of the theory. As a result, only for gauge groups that contain a $SU(n)$, $n \geq 3$, or a $U(1)$ factor, there can be a non-trivial tensor $d_{\Sigma\Lambda\Omega}$. The second important ingredient that determines the precise value of $d_{\Sigma\Lambda\Omega}$ is the fermion content. Even for gauge groups with $SU(n)$ or $U(1)$ factors, it is still possible to obtain a vanishing tensor $d_{\Sigma\Lambda\Omega}$ by carefully tuning the representations of the fermions. For example, this is the case in the Standard Model, where the total gauge group is given by $U(1) \times SU(2) \times SU(3)$, but the properties of the chiral fermions are such that all $d_{\Sigma\Lambda\Omega}$ vanish. Therefore the SM is an example of a perfectly consistent quantum theory.

In principle, the reader has now enough information to understand the essence of our upcoming discussion in chapters 5 and 6 about anomaly cancellation in certain supergravity theories. Therefore, if one does not pursue a detailed understanding, the remainder of this section can be safely skipped. For us, however, it is also important to be able to obtain a precise expression for the anomaly in the relevant supergravity theories. Since there are several complications such as a scalar-dependent metric in the kinetic terms of the fermions, the perturbative approach is less suited and we need to introduce some more powerful machinery. This will be the path integral formalism, which is the subject of §2.4 and §2.4.

### Path integral method

In contrast to our discussion in the previous section, the path integral method involves the Lagrangian as a fundamental quantity[14], and it does not mention operators at all. Instead, a special kind of integral over classical fields is employed. The quantum properties of a system appear because the motion of a particle between two points can proceed via an infinite variety of classical trajectories and each of these alternatives has a certain contribution to the total transition amplitude. This results in a path integral from which, at least in principle, all properties of a system can be deduced by using functional techniques.

Without going into the details of the construction, the $n$-point correlation functions (2.68) have the following path integral representation:

$$G^{(n)}(x_1,\ldots,x_n) = N\int[\mathrm{d}\Phi]\,\Phi(x_1)\ldots\Phi(x_n)\exp\left[\frac{\mathrm{i}}{\hbar}\int \mathrm{d}^4x\,\mathcal{L}(\Phi,\partial\Phi)\right]. \tag{2.76}$$

Here, $\int[\mathrm{d}\Phi]$ is a functional integral over all possible field configurations that satisfy certain boundary conditions. The measure $[\mathrm{d}\Phi]$ will play an important role in our discussion of quantum anomalies in §2.4. $N$ is a normalization factor, its inverse is given by

$$N^{-1} = \int[\mathrm{d}\Phi]\exp\left[\frac{\mathrm{i}}{\hbar}\int \mathrm{d}^4x\,\mathcal{L}(\Phi,\partial\Phi)\right]. \tag{2.77}$$

To emphasize the quantum nature of the path integral, we have written an explicit factor of $\hbar$ in (2.76) and (2.77), although in future formulae we will omit this factor.

In general, the vacuum expectation value of any time ordered combination of operators $\hat{\mathcal{O}}_1\ldots\hat{\mathcal{O}}_n$ can be computed via

$$\langle\hat{\mathcal{O}}_1\ldots\hat{\mathcal{O}}_n\rangle = \int[\mathrm{d}\Phi]\,\mathcal{O}_1\ldots\mathcal{O}_n\exp\left[\mathrm{i}\int \mathrm{d}^4x\,\mathcal{L}(\Phi,\partial\Phi)\right]. \tag{2.78}$$

In the case of the $n$-point correlation function (2.76), for example, we have $\mathcal{O}_i = \Phi(x_i)$. However, the operators we are again most interested in are of the form $\hat{\mathcal{O}} = \hat{J}^\mu$, with $J^\mu$ a classical symmetry current. Let us investigate the quantum properties of these currents using the path integral approach.

### Anomalies and the path integral formalism

From our treatment of anomalies in §2.4, we know that they arise from chiral couplings between fermions and gauge fields. Therefore, we will require the

[14]Indeed, in equation (2.68) the Hamiltonian enters as a fundamental quantity. It appears both in the Heisenberg operators,

$$\hat{\Phi}(x) = \mathrm{e}^{\mathrm{i}H(t-t_0)}\hat{\Phi}(t_0,\boldsymbol{x})\mathrm{e}^{-\mathrm{i}H(t-t_0)}\,, \tag{2.75}$$

and less obviously, in the definition of the vacuum $|\Omega\rangle$.

presence of these fields in the path integral, i.e. $\Phi \in \{A_\mu, \psi, \bar{\psi}\}$. In addition, there can be extra fields. For example, in generic supergravity theories there are couplings to scalar particles. In order to keep our discussion as simple as possible, we will not explicitly take these extra fields into account, although the formalism can be extended straightforwardly. We also assume the presence of a classical Lagrangian $\mathcal{L}_{\rm int}[\psi, \bar{\psi}, A_\mu] = \mathcal{L}_{\rm matter} + A_\mu{}^\Sigma J^\mu{}_\Sigma$, with $J^\mu{}_\Sigma$ a current similar to (2.72). This Lagrangian is invariant under a set of chiral gauge transformations with local parameters $\theta^\Lambda(x)$, as described in §2.4.

Our main object of interest will be the so-called "effective action" $\Gamma[A]$, which is defined as follows:

$$\mathrm{e}^{\mathrm{i}\Gamma[A]} \equiv \int [\mathrm{d}\psi \mathrm{d}\bar{\psi}]\, \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\, \mathcal{L}_{\rm int}(\psi,\bar{\psi},A)} \,. \tag{2.79}$$

The main difference between the right hand side of this expression, and the inverse normalization factor $N^{-1}$ in (2.77), is that we have not yet included the functional integral over the gauge fields $A_\mu{}^\Sigma(x)$. Therefore, the gauge fields are said to be "external". Of course, if one wants to calculate the vacuum expectation values, the integral over the vectors still needs to be done, i.e.,

$$N^{-1} = \int [dA] \mathrm{e}^{\mathrm{i}S[A]} \quad \text{with} \quad S[A] = -\frac{1}{4}\int \mathrm{d}^4x\, \mathcal{F}_{\mu\nu}{}^\Lambda \mathcal{F}^{\mu\nu\Lambda} + \Gamma[A]\,, \tag{2.80}$$

and similar for (2.78). If $\Gamma[A]$ is gauge invariant, so is $S[A]$ and everything is consistent: the unphysical degrees of freedom can be eliminated via the gauge fixing procedure and one ends up with a renormalizable and unitary quantum theory. However, if $\Gamma[A]$ is not gauge invariant, all this breaks down, and the theory develops a quantum anomaly.

In order to characterize the properties of the anomaly, we need to find out when (and how) $\Gamma[A]$ violates gauge invariance. From (2.79), it is easy to pinpoint the problem. Indeed, consider the following equalities:

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\Gamma[A']} = \int [\mathrm{d}\psi \mathrm{d}\bar{\psi}]\, \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\, \mathcal{L}_{\rm int}(\psi,\bar{\psi},A')} \quad &= \quad \int [\mathrm{d}\psi' \mathrm{d}\bar{\psi}']\, \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\, \mathcal{L}_{\rm int}(\psi',\bar{\psi}',A')} \\ &= \quad \int [\mathrm{d}\psi' \mathrm{d}\bar{\psi}']\, \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\, \mathcal{L}_{\rm int}(\psi,\bar{\psi},A)} \end{aligned} \tag{2.81}$$

The first equality is the definition (2.79). In the second equality, we made a change of basis, where $\psi'$ and $\bar{\psi}'$ denote the gauge transformed fermions. The last equality is a consequence of the gauge invariance of the action. Therefore, only when the fermion measure $[\mathrm{d}\psi \mathrm{d}\bar{\psi}]$ is not gauge invariant, the effective action $\Gamma[A]$ transforms non-trivially under gauge transformations, i.e.,

$$[\mathrm{d}\psi \mathrm{d}\bar{\psi}] \neq [\mathrm{d}\psi' \mathrm{d}\bar{\psi}'] \quad \Leftrightarrow \quad \Gamma[A] \neq \Gamma[A']\,. \tag{2.82}$$

Let us introduce the following notation for the Jacobian of the transformation:

$$[\mathrm{d}\psi' \mathrm{d}\bar{\psi}'] \equiv [\mathrm{d}\psi \mathrm{d}\bar{\psi}]\mathcal{J} = [\mathrm{d}\psi \mathrm{d}\bar{\psi}] \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\, \theta^\Sigma(x) \mathcal{A}_\Sigma(x)}\,, \tag{2.83}$$

with $\mathcal{A}_\Sigma(x)$ a local function of the fields that will momentarily be related to the expression for the anomaly (hence our suggestive notation). This notation can now be used to complete the calculation in equation (2.81):

$$\mathrm{e}^{\mathrm{i}\Gamma[A']} = \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\, \theta^\Sigma(x)\mathcal{A}_\Sigma(x)}\mathrm{e}^{\mathrm{i}\Gamma[A]}\,. \tag{2.84}$$

Therefore, an infinitesimal gauge transformation of the effective action takes the form

$$\delta(\theta)\Gamma[A] = \int \mathrm{d}^4x\, \theta^\Sigma(x)\mathcal{A}_\Sigma(x)\,. \tag{2.85}$$

On the other hand, for any functional $\Gamma[A]$ of the gauge fields only, its gauge variation is given by

$$\begin{aligned}
\delta(\theta)\Gamma[A] &= \int \mathrm{d}^4x\, \frac{\delta\Gamma[A]}{\delta A_\mu{}^\Sigma(x)}\delta A_\mu{}^\Sigma(x) = \int \mathrm{d}^4x\, \frac{\delta\Gamma[A]}{\delta A_\mu{}^\Sigma(x)}\left(D_\mu\theta(x)\right)^\Sigma \\
&= -\int \mathrm{d}^4x\, \theta^\Sigma(x)\left(D_\mu \frac{\delta\Gamma[A]}{\delta A_\mu{}^\Sigma(x)}\right)\,,
\end{aligned} \tag{2.86}$$

where in the last step we used the fact that one can do a partial integration with the covariant derivative as if it were just an ordinary derivative. If we then combine (2.85) and (2.86), we obtain the well-known relation

$$D_\mu \frac{\delta\Gamma[A]}{\delta A_\mu{}^\Sigma(x)} = D_\mu\langle J^\mu{}_\Sigma(x)\rangle_A = -\mathcal{A}_\Sigma(x)\,. \tag{2.87}$$

The first equality requires some explanation. From our form of the action $\mathcal{L}_{\text{int}}$ above equation (2.79), we know that the gauge fields $A_\mu{}^\Sigma$ couple linearly to the gauge currents $J^\mu{}_\Sigma$. In a sense, the gauge fields can therefore be interpreted as sources for the currents. If we then take the functional derivative of $\Gamma[A]$ with respect to the sources, we obtain

$$\begin{aligned}
\frac{\delta\Gamma[A]}{\delta A_\mu{}^\Sigma(x)} &= \mathrm{e}^{-\mathrm{i}\Gamma[A]}\frac{\delta}{\delta A_\mu{}^\Sigma(x)}\int [\mathrm{d}\psi \mathrm{d}\bar{\psi}]\mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\left[\mathcal{L}_{\text{matter}}+A_\mu{}^\Sigma J^\mu{}_\Sigma\right]} \\
&= \mathrm{e}^{-\mathrm{i}\Gamma[A]}\int [\mathrm{d}\psi \mathrm{d}\bar{\psi}]J^\mu{}_\Sigma(x)\mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\left[\mathcal{L}_{\text{matter}}+A_\mu{}^\Sigma J^\mu{}_\Sigma\right]} \\
&= \langle J^\mu{}_\Sigma(x)\rangle_A\,,
\end{aligned} \tag{2.88}$$

where the subscript $A$ reminds us that the expectation value is computed with fixed external gauge fields. Finally, with the help of these identifications and keeping §2.4 in mind, the function $\mathcal{A}_\Sigma(x)$ in (2.87) can be identified with the quantum anomaly.

In order to compute $\mathcal{A}_\Sigma(x)$ using the path integral formalism, it suffices to single out the Jacobian from the transformation of the measure in (2.83). However, more powerful methods exist that follow immediately from (2.85). Among these is the reformulation in terms of a BRST cohomology. We refer to [24] for an introduction to these methods.

## 2.5 Things to remember

We have now come to the end of this lengthy chapter that was intended to familiarize the reader with the most important concepts of gauge theories. We started from the easiest example which is Maxwell theory, and extended our analysis to generic non-Abelian Yang-Mills theories. Along the way, we discovered a powerful mechanism to construct these theories, namely the "gauging procedure". Because this mechanism will be at the basis of most of our results in this thesis, let us once more summarize the different steps:

1. One starts from a matter theory with a *global* symmetry group $G$.
2. One promotes (part of) the global transformations to *local* gauge transformations with a gauge group $G_{\text{local}}$. The theory will be modified such that the new Lagrangian is invariant under these local transformations.
3. This requires the construction of *covariant derivatives* through the introduction of $\dim_{G_{\text{local}}}$ gauge potentials $A_\mu{}^\Sigma$. In general, the potentials transform under gauge transformations with the derivative of the local gauge parameters.
4. Finally, one obtains the fully interacting Lagrangian ($\mathcal{L}_{\text{YM}}$) by substituting a covariant derivative for every ordinary space-time derivative in the original Lagrangian, and by adding the free-boson Lagrangian ($\mathcal{L}_A$).

This procedure is not restricted to internal symmetries, though. Indeed, in the next chapter we will see how gravity, and more generally supergravity, arise as the gauge theories of Poincaré symmetry and supersymmetry respectively.

Besides the classical aspects of a gauge theory, we have also investigated some of their quantum properties. We highlighted one aspect, which is the possible presence of quantum anomalies. Such anomalies signal the breakdown of a classical symmetry in the quantum regime. Since anomalous gauge symmetries prevent a good quantum behaviour, we investigated the constraints on the type of gauge groups and on the field content, in order to have an anomaly-free theory. We found the following interplay between these two aspects: only theories with a chiral fermion content and a gauge group that has at least one $SU(n)$, $n \geq 3$ or $U(1)$ factor, can be anomalous. Unfortunately, these are physically also the most interesting cases. Therefore, one also has to tune the precise representation content (i.e., the charges) of the chiral fermions, such that ultimately, the anomaly vanishes. In nature, such a possibility is explicitly realised via the Standard Model. In this thesis, anomalies will appear in chapters 5 and 6, where we will discuss an alternative mechanism to obtain anomaly-freedom of more general (supergravity) theories.

Chapter 3

# Supersymmetry and supergravity

In the previous chapter we have formulated the gauge principle, which provides a series of well-defined steps to relate the global symmetries of a free matter theory to local invariances of an interacting theory. The difference between both theories is contained in the minimal coupling of the matter fields to certain vector potentials that describe the force-carrying particles.

Whereas our discussion has been focused on *internal* symmetries so far, we will now shift our attentions towards *space-time* symmetries. The best known example is a global Poincaré transformation, but also global supersymmetry falls into this category. After the initial discussion about their global properties, we will move on again, and study the physical aspects of their corresponding gauge theories. In this way, we are naturally led to gravity as the gauge theory for Poincaré symmetry, and supergravity as the gauge theory for supersymmetry. Our principle objective is to give the reader some insight into the general structure of supergravity theories, and their coupling to general matter fields. The knowledge of these couplings will be particularly relevant for the discussion in chapter 4 and the presentation of our results in chapters 5, 6 and 7.

The outline of this chapter is as follows. In §3.1 we will review some aspects of global and local Poincaré transformations. Amazingly, we recover gravity as a gauge theory with the graviton as its associated gauge field. This way of introducing gravity is probably not familiar to all readers, but it does facilitate the transition to supergravity later on. Once more, this first section should also be seen as an opportunity for the reader to become familiar with several useful concepts and notations. Section 3.2 contains a short introduction to global supersymmetry. Since this is also the subject of numerous excellent textbooks and reviews [26–30], we will only consider the aspects that are relevant for us. Our focus will be on four space-time dimensions with one extra supersymmetry generator. Finally in §3.3 we introduce the basic ingredients of four-dimensional pure supergravity and its coupling to various matter multiplets.

## 3.1 Invitation: gravity as a gauge theory

In the absence of gravity, the laws of physics reduce to those of special relativity. The structure of space-time is that of a flat Minkowski space, with a metric $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. As a result, the equations that govern the physical laws should be covariant under global Poincaré transformations.

### Global Poincaré symmetries

The Poincaré group consists of space-time translations $x^\mu \to x^\mu - a^\mu$ and Lorentz rotations $x^\mu \to \Lambda^{-1\mu}{}_\nu x^\nu$, with $a^\mu$ a constant vector and $\Lambda^\mu{}_\nu$ matrices that satisfy $\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$. Its algebra is spanned by the generators $P_\mu$ for translations and $M_{\mu\nu} = -M_{\nu\mu}$ for Lorentz rotations. The non-zero commutation relations are

$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu[\rho} M_{\sigma]\nu} - \eta_{\nu[\rho} M_{\sigma]\mu} \,, \tag{3.1}$$

$$[P_\mu, M_{\nu\rho}] = \eta_{\mu[\nu} P_{\rho]} \,. \tag{3.2}$$

The fields that make up the theory transform in irreducible representations of the Poincaré group. If we denote a generic field with $\Phi^i$, where $i$ is an index that labels the components of the representation, then $\Phi^i$ transforms as follows under general Poincaré transformations:

$$\delta\Phi^i = \left[a^\mu (P_\mu)^i{}_j - \lambda^{\rho\sigma} (M_{\rho\sigma})^i{}_j\right] \Phi^j \,, \tag{3.3}$$

with $a^\mu$ and $\lambda^{\rho\sigma}$ small constant parameters. The action of the translation generator on the fields is easy; for all types of indices $i, j, \ldots$, we have

$$(P_\mu)^i{}_j = \delta^i_j \partial_\mu \,. \tag{3.4}$$

The action of the Lorentz generators is more complicated and depends on the type of field we are considering:

$$\text{scalars:} \qquad M_{\rho\sigma}\phi = x_{[\rho}\partial_{\sigma]}\phi\,, \tag{3.5}$$

$$\text{fermions:} \qquad (M_{\rho\sigma})_\alpha{}^\beta\psi_\beta = \left[\delta_\alpha^\beta x_{[\rho}\partial_{\sigma]} + \tfrac{1}{4}(\gamma_{\rho\sigma})_\alpha{}^\beta\right]\psi_\beta\,, \tag{3.6}$$

$$\text{vectors:} \qquad (M_{\rho\sigma})^\mu{}_\nu V^\nu = \left[\delta_\nu^\mu x_{[\rho}\partial_{\sigma]} + \delta^\mu_{[\rho}\eta_{\sigma]\nu}\right]V^\nu\,, \tag{3.7}$$

where $\alpha, \beta, \ldots$ in (3.6) are spinor indices.[1] The first term in (3.5)-(3.7) is identical for each of the fields and it has an explicit dependence on the coordinates.[2] The other terms in (3.6) and (3.7) describe the transformation of the fields as elements of the representation space; spinors transform with a gamma matrix and vectors with a metric. In the remainder of this text, these extra contributions will be denoted by $(L_{\rho\sigma})^i{}_j$, for example

$$(L_{\rho\sigma})_\alpha{}^\beta\psi_\beta = \frac{1}{4}(\gamma_{\rho\sigma})_\alpha{}^\beta\psi_\beta\,. \tag{3.8}$$

Finally, one can check that for each of the $(M_{\rho\sigma})^i{}_j$ above, and for $(P_\mu)^i{}_j$ in (3.4), the commutation relations (3.1) and (3.2) are indeed satisfied.

The dynamics of the matter fields is governed by a Lagrangian $\mathcal{L}_\mathrm{m}[\Phi, \partial\Phi]$ that should transform as a scalar under the variations (3.3):

$$\delta\mathcal{L}_\mathrm{m} = \xi^\mu\partial_\mu\mathcal{L}_\mathrm{m}\,, \qquad \text{with} \quad \xi^\mu \equiv a^\mu + \lambda^{\mu\nu}x_\nu\,. \tag{3.9}$$

Only then is the action $S_\mathrm{m} = \int \mathrm{d}^4x\,\mathcal{L}_\mathrm{m}$ invariant under Poincaré transformations (remark that $\partial_\mu\xi^\mu = 0$).

### Local Poincaré transformations

Let us now consider what happens when we promote the global transformations (3.3) to *local* gauge transformations with space-time dependent parameters $a^\mu(x)$ and $\lambda^{\rho\sigma}(x)$. In the spirit of the previous chapter, we expect to find new gauge fields that describe the mediation of a new force.

In order to get some intuition about the results of this gauging procedure, let us study the form of the Poincaré transformations (3.3) in more detail. We note that $\delta\Phi^i$ can be rewritten as follows:

$$\begin{aligned}\delta\Phi^i &= \left[a^\mu\delta^i_j\partial_\mu - \lambda^{\rho\sigma}\delta^i_j x_{[\rho}\partial_{\sigma]} - \lambda^{\rho\sigma}(L_{\rho\sigma})^i{}_j\right]\Phi^j \\ &= \xi^\mu\partial_\mu\Phi^j - \lambda^{\rho\sigma}(L_{\rho\sigma})^i{}_j\Phi^j\,. \end{aligned}\tag{3.10}$$

[1]For more information and our conventions we refer to appendix A.

[2]This is the main feature that distinguishes space-time transformations form the internal symmetries that were discussed in the previous chapter.

So far, the vector $\xi^\mu$ has a particular (linear) dependence on the coordinate $x^\mu$. However, if we promote the parameters $a^\mu$ and $\lambda^{\rho\sigma}$ to local functions, the vector $\xi^\mu$ becomes an arbitrary function of space and time. Then the first term in (3.10) describes the transformation of the fields $\Phi^i$ under a general coordinate transformation.

Apparently, for a Poincaré group with local parameters, part of the transformation of the fields becomes a general coordinate transformation. Demanding invariance under these transformations corresponds to the "principle of general covariance" in general relativity. Therefore, we expect to find gravity as a new force in our construction.

We will now investigate this claim in more detail, by carefully considering the different steps in the gauging procedure (see chapter 2). But before we do that, let us settle some notational issue. From now on we will always use the Roman alphabet $a, b, \ldots$ to indicate "flat indices", and Greek letters $\mu, \nu, \ldots$ for curved space-time indices. All the objects that make reference to a local frame therefore carry indices $a, b, \ldots$. The two sets of indices will be related by the vierbein, to be introduced in due course.

The gauging procedure consists of the following steps:

**Choice of parameter basis.** The parameters of local translations and local Lorentz transformations are $\xi^\mu(x)$ and $\lambda^{\mu\nu}(x)$ respectively. Remark that we have performed a change of basis from the original parametrization $a^\mu(x)$ and $\lambda^{\mu\nu}(x)$ to the new basis of independent transformations $\xi^\mu(x)$ and $\lambda^{\mu\nu}(x)$. Under these transformations, the scalar fields $\phi(x)$, fermions $\psi_\alpha(x)$ and vectors $V^\mu(x)$ transform as follows:

$$\delta(\xi,\lambda)\phi(x) = \xi^\mu(x)\partial_\mu\phi(x)\,, \tag{3.11}$$

$$\delta(\xi,\lambda)\psi_\alpha(x) = \xi^\mu(x)\partial_\mu\psi_\alpha(x) - \frac{1}{4}\lambda^{\mu\nu}(x)(\gamma_{\mu\nu})_\alpha{}^\beta\psi_\beta\,, \tag{3.12}$$

$$\delta(\xi,\lambda)V^\mu(x) = \xi^\nu(x)\partial_\nu V^\mu(x) - (\partial_\nu\xi^\mu(x))V^\nu\,. \tag{3.13}$$

**Covariant derivatives.** In order to construct a theory that is invariant under these local transformations, ordinary space-time derivatives of the fields need to be replaced by covariant derivatives. Under local transformations, these should transform in the same way as the $\partial_\mu\Phi^i$ transform under global transformations, i.e.

$$\delta(\xi,\lambda)D_\mu\Phi^i = \xi^\nu(x)\partial_\nu\left(D_\mu\Phi^i\right) + (\partial_\mu\xi^\nu(x))D_\nu\Phi^i - \lambda^{\rho\sigma}(x)(L_{\rho\sigma})^i{}_j\left(D_\mu\Phi^j\right). \tag{3.14}$$

This can be achieved via the following construction:

$$D_\mu \Phi^i(x) \equiv \left(\delta^i_j \partial_\mu + \omega_\mu{}^{bc}(x)(L_{bc})^i{}_j\right) \Phi^j(x)\,, \tag{3.15}$$

together with an appropriate local transformation of $\omega_\mu{}^{bc}(x)$ (see (3.18)). The new field $\omega_\mu{}^{bc}(x)$, called the spin connection, is the gauge field associated to the Lorentz operators $M_{bc}$. It carries one space-time index and the antisymmetric combination of two flat indices. The latter are defined with respect to a local reference frame that is fixed by the vierbein. This object, denoted by $e_a{}^\mu(x)$, is an orthogonal matrix with inverse $e_\mu{}^a(x)$ ($e_a{}^\mu e_\mu{}^b = \delta^b_a$ and $e_a{}^\mu e_\nu{}^a = \delta^\mu_\nu$), and it will be used to translate between flat and curved indices as follows:

$$T^\mu = e_a{}^\mu T^a\,, \qquad T_\mu = e_\mu{}^a T_a\,. \tag{3.16}$$

In particular, we can define a covariant derivative with a flat index as

$$D_a \Phi^i \equiv e_a{}^\mu D_\mu \Phi^i\,. \tag{3.17}$$

Under local transformations, $D_a\Phi^i$ should also transform in the same way as $\partial_a \Phi^i$ transforms under global transformations. In order to satisfy this requirement, together with the equation (3.14), one has to impose the following local transformations of the gauge fields:

$$\delta(\xi,\lambda)\omega_\mu{}^{ab} \;=\; \xi^\nu \partial_\nu \omega_\mu{}^{ab} + (\partial_\mu \xi^\nu)\omega_\nu{}^{ab} + \partial_\mu \lambda^{ab} - 2\lambda_c{}^{[a}\omega_\mu{}^{b]c}\,, \tag{3.18}$$

$$\delta(\xi,\lambda)e_\mu{}^a \;=\; \xi^\nu \partial_\nu e_\mu{}^a + (\partial_\mu \xi^\nu)e_\nu{}^a - \lambda^a{}_b e_\mu{}^b\,. \tag{3.19}$$

The structure of these transformations is not surprising. The first two terms in each equation form the Lie derivative with respect to $\xi^\nu$, for example:

$$\mathcal{L}_\xi e_\mu{}^a = \xi^\nu \partial_\nu e_\mu{}^a + (\partial_\mu \xi^\nu)e_\nu{}^a\,. \tag{3.20}$$

This is the expected transformation under general coordinate transformations of an object with a coordinate index $\mu$. The other terms are the local Lorentz transformations; for $e_\mu{}^a$ this corresponds to $-\lambda^{cd}(L_{cd})^a{}_b e_\mu{}^b$. For $\omega_\mu{}^{ab}$ there are two parts. The third term in (3.18) is the derivative on the gauge parameter $\lambda^{ab}$, which is characteristic for the transformation of a gauge field. The last term corresponds to $2\lambda^{de}(L_{de})^{[a}{}_c\omega_\mu{}^{b]c}$, again as expected.

**Covariant matter action.** Once the ordinary space-time derivatives have been replaced by covariant ones, there is still one subtlety that remains. This can be seen from (3.9). The modified Lagrangian $\mathcal{L}[\Phi, D\Phi]$ transforms as $\delta\mathcal{L}_{\rm m} = \xi^\mu(x)\partial_\mu\mathcal{L}_{\rm m}$, but this time the right hand side is not a total derivative. Therefore the action is not yet invariant. To cure this problem, we will multiply $\mathcal{L}_{\rm m}[\Phi, D\Phi]$ by an arbitrary function $\Lambda(e,\omega)$ of the fields $e_a{}^\nu$ and $\omega_\mu{}^{ab}$, such that

$$\delta(\xi,\lambda)\left(\Lambda\mathcal{L}_{\rm m}\right) = \partial_\mu\left(\xi^\mu \Lambda \mathcal{L}_{\rm m}\right)\,. \tag{3.21}$$

This leads to the condition $\delta(\xi,\lambda)\Lambda = \partial_\mu(\Lambda\xi^\mu)$, which has a solution $\Lambda(e,\omega) = \det(e_\mu{}^a) \equiv e$. Then the action

$$S_{\rm m} = \int \mathrm{d}^4x\, e\mathcal{L}_{\rm m}[\Phi, D\Phi] \tag{3.22}$$

is invariant under local translations and local Lorentz transformations.

**Einstein-Hilbert action.** In the last step, we still need to construct an action that describes the "free" gauge fields. The "field strengths" can be constructed using the equivalent of (2.54),

$$[D_a, D_b]\Phi^i = -R_{ab}{}^c(P_c)^i{}_j\Phi^j + R_{ab}{}^{cd}(L_{cd})^i{}_j\Phi^j \tag{3.23}$$

We find

$$R_{ab}{}^c = (e_a{}^\mu e_b{}^\nu - e_a{}^\nu e_b{}^\mu)\, R_{\mu\nu}{}^c\,, \qquad R_{ab}{}^{cd} = (e_a{}^\mu e_b{}^\nu - e_a{}^\nu e_b{}^\mu)\, R_{\mu\nu}{}^{cd} \tag{3.24}$$

and

$$R_{\mu\nu}{}^c \;=\; \partial_{[\mu}e_{\nu]}{}^c + \omega_{[\mu}{}^{cd}e_{\nu]d}\,, \tag{3.25}$$

$$R_{\mu\nu}{}^{cd} \;=\; \partial_{[\mu}\omega_{\nu]}{}^{cd} + \omega_{[\mu}{}^{ec}\omega_{\nu]}{}^{df}\eta_{ef}\,. \tag{3.26}$$

In the first order formalism, one constructs an action that has both the vierbein $e_\mu{}^a$ and spin connection $\omega_\mu{}^{ab}$ as its independent variables

$$S_{\rm EH} = \int \mathrm{d}^4x\, \frac{1}{2\kappa^2} e\left(e_a{}^\mu e_b{}^\nu R_{\mu\nu}{}^{ab}(\omega)\right)\,, \tag{3.27}$$

with $\kappa^2 = 8\pi G_N/c^4$ a constant that will be set to one in the following. The equation of motion for the spin connection is given by $R_{\mu\nu}{}^c = 0$. This equation can be solved for the spin connection in terms of the vierbein[3], $\omega_\mu{}^{ab} = \omega_\mu{}^{ab}(e)$, and when this result is substituted into the field equation for the vierbein, one recovers the usual Einstein equations. Therefore, from the equivalent action

$$S_{\rm EH} = \int \mathrm{d}^4x\, \frac{1}{2\kappa^2} e\left[e_a{}^\mu e_b{}^\nu\left(\partial_{[\mu}\omega_{\nu]}{}^{cd}(e) + \omega_{[\mu}{}^{ec}(e)\omega_{\nu]}{}^{df}(e)\eta_{ef}\right)\right] \tag{3.28}$$

one obtains the correct dynamics via Euler-Lagrange variation with respect to the vierbein only. This is the second order formalism which is completely equivalent to the conventional formulation of general relativity in terms of the metric $g_{\mu\nu}$ and

[3] When gravity is coupled to spinor fields, such as in supergravity, the equation of motion for the spin connection contains terms that depend on the spinors. Then its solution can be written as $\omega_\mu{}^{ab} = \omega_\mu{}^{ab}(e) + T_\mu{}^{ab}$, with $T_\mu{}^{ab}$ a contortion tensor that is bilinear in the spinor fields.

Levi-Civita connection $\Gamma^{\rho}_{\mu\nu}(g)$. The two formulations are related via the defining relation of the vierbein, $g_{\mu\nu} = e_\mu{}^a \eta_{ab} e_\nu{}^b$.

This concludes our construction of general relativity as a gauge theory for the Poincaré symmetry algebra. The total action for the coupling of matter fields to gravity is given by

$$S = S_{\mathrm{m}} + S_{\mathrm{EH}}\,, \tag{3.29}$$

and the energy-momentum tensor of the theory can be determined from $\mathcal{L}_{\mathrm{m}}$. We will now take this result as a starting point for the introduction of supergravity, which is the gauge theory corresponding to an extension of the Poincaré algebra, namely the Poincaré superalgebra. In §3.2 we first discuss the general properties of this superalgebra as a global space-time symmetry algebra.

## 3.2 Global supersymmetry

Supersymmetry was identified by R. Haag, J.T. Łopuszanski and M. Sohnius [4] as the only space-time symmetry (in addition to Poincaré transformations) that allows for non-trivial scattering of particles. Our first goal is to review these results, and discuss the formal consequences of space-time supersymmetry at the level of classical field theories. Our second goal is to introduce a supersymmetric theory of gravity. This will lead us to the concept of supergravity, which will be the topic of §3.3 and the framework for most of our discussions in later chapters of this text. We point out to the reader that we only pursue a formal development of these classical theories, without studying interesting aspects such as their solutions, possible quantum effects, or their relation to string theory.

### Supersymmetry algebra

Haag, Łopuszanski and Sohnius presented in their paper a classification of the global space-time symmetries of a relativistic field theory that are compatible with having non-trivial scattering amplitudes. Their results can be summarized as follows:[4]

| | | |
|---|---|---|
| Bosonic generators: | - translations: | $P_a$ |
| | - Lorentz rotations: | $M_{ab}$ |
| | - global internal (R-)symmetries: | $T_A$ |
| Fermionic generators: | - supersymmetry: | $Q$ |

[4] In the absence of massive particles, the set of bosonic generators can be extended to contain all conformal generators. This result was already obtained by Coleman and Mandula in [31].

Remarkably, besides the original bosonic (or "even") generators, they also found the possibility of fermionic (or "odd") symmetry generators. The latter were baptized "supersymmetry" generators, and as we pointed out in the introduction, they have played a major role in high energy physics ever since. Supersymmetry invariance relates the bosonic and fermionic particle content of physical theories, and it puts strong restrictions on the form of their interaction terms. Both aspects will be discussed briefly in §3.2 and §3.2. But before we come to that, let us review some of the properties of the odd generators, as well as the structure of the Poincaré superalgebra.

The operators $Q$ are anticommuting spinors that turn bosonic states into fermionic states, and vice versa,

$$Q|\text{Boson}\rangle = |\text{Fermion}\rangle\,, \qquad \text{Q}|\text{Fermion}\rangle = |\text{Boson}\rangle\,. \tag{3.30}$$

If we restore all the indices, we have $\mathcal{N}$ generators $Q^i_\alpha$ with $\alpha$ a space-time spinor index and $i = 1, \ldots, \mathcal{N}$ an index that labels the different supercharges. For $\mathcal{N} = 1$ we say the theory is minimally supersymmetric, whereas for $\mathcal{N} > 1$ it has extended supersymmetry. Moreover, there is an upper bound on the possible values of $\mathcal{N}$, which is determined by the number of space-time dimensions we are working in. We will see that in 4 dimensions, $\mathcal{N} \leq 4$.

Since our discussion is restricted to 4 dimensions, we can split the spinors into a left handed and right handed chirality part, which we indicate by the position of the $i$ index:

$$\frac{1}{2}\left(1-\gamma_5\right)_\alpha{}^\beta Q^i_\beta = Q^i_\alpha\,, \qquad \frac{1}{2}\left(1+\gamma_5\right)_\alpha{}^\beta Q_{\beta i} = Q_{\alpha i}\,. \tag{3.31}$$

The supersymmetry generators nicely combine with the original Poincaré generators into a new algebraic structure, called the Poincaré superalgebra. It is built by both commutators and anticommutators of bosonic and fermionic generators in the following schematic way:

$$[B, B] = B, \qquad [B, F] = F, \qquad \{F, F\} = B\,. \tag{3.32}$$

The precise form of the (anti)commutators was fixed by Haag, Łopuszanski and Sohnius. On top of the original Poincaré commutation relations (3.1) and (3.2), they found the following non-vanishing relations that involve the $Q^i_\alpha$ and $Q_{\alpha i}$:

$$\begin{aligned}
[M_{ab}, Q^i_\alpha] &= -\tfrac{1}{4}(\gamma_{ab})_\alpha{}^\beta Q^i_\beta\,, \\
[T_A, Q^i_\alpha] &= (U_A)^i{}_j Q^j_\alpha\,, \qquad [T_A, Q_{\alpha i}] = (U_A)_i{}^j Q_{\alpha j}\,, \\
\{Q^i_\alpha, Q_{\beta j}\} &= \left(\gamma^a \mathcal{C}^{-1}\right)_{\alpha\beta} P_a \delta^i_j\,, \\
\{Q_{\alpha i}, Q_{\beta j}\} &= \left(\mathcal{C}^{-1}\right)_{\alpha\beta} \omega^M_{ij} Z_M\,.
\end{aligned} \tag{3.33}$$

The first commutator reflects the fact that $Q^i_\alpha$ transforms in the spinor representation under Lorentz transformations. In the second line, the rotation of the supercharges under $R$-symmetry is given. In 4 dimensions, $(U_A)^i{}_j$ is a unitary matrix that belongs to $U(\mathcal{N})$, and $(U_A)_i{}^j = (U_A)^j{}_i$. The third line contains the defining relation for the supersymmetry generators: they square to translations. This relation has several interesting consequences, such as an equal number of bosonic and fermionic states in each realization of the supersymmetry algebra. We will come back to this feature in the next section. Finally, in the last line of (3.33) we have included the option of "central charges", denoted by $Z_M$. Although central charges are essential for several applications due to the BPS bound they impose, we will not consider them any further in this text.

This concludes our discussion about the general properties of the supercharges and the Poincaré superalgebra. We will now proceed with the construction of explicit realizations of this algebra, in order to describe the different particles that are present in supersymmetric theories.

## Supermultiplets

For every Poincaré superalgebra there exist a number of irreducible representations that form the "single-particle" states of a supersymmetric theory. These states are called supermultiplets, and each multiplet is a collection of ordinary bosonic and fermionic fields, which are known as "superpartners" of each other. Recall from (3.30) that superpartners are related via supersymmetry transformations. Since the number of supersymmetry generators is finite and given by $\mathcal{N}$, also the number of fermions and bosons in each multiplet is restricted. Table 3.1 contains all the massless irreducible representations[5] of the superalgebra in $4D$ (we write their on-shell field content).

**Table 3.1:** Supersymmetry multiplets in $4D$

| $\mathcal{N}$ | $\sharp$ supercharges | multiplets & field content |
|---|---|---|
| 1 | 4 | chiral: $(z,\, \chi_{(L)})$ |
| | | vector: $(A_\mu,\, \lambda_{(R)})$ |
| 2 | 8 | hyper: $(q^X,\, \zeta^A)$, $\quad X = 1,\ldots,4;\, A = 1,2$ |
| | | vector: $(A_\mu,\, \Omega^i,\, z)$, $\quad i = 1,2$ |
| 4 | 16 | vector: $(A_\mu,\, \psi^j,\, \phi_{jk})$, $\quad j,k = 1,\ldots,4$ |

[5] Besides the irreducible representations, there are also various reducible multiplets. These include the real multiplet in $\mathcal{N} = 1$ and the chiral multiplet in $\mathcal{N} = 2$. Both have the property that they reduce to the vector multiplet upon elimination of some of their components via a "supersymmetric gauge fixing".

The vector multiplets have a spin-1 vector field $A_\mu(x)$ as their highest component. The vector is supplemented by chiral spinors as its supersymmetric partners. In $\mathcal{N}=1$ there is only 1 right-handed spinor, denoted by $\lambda_{(R)}(x)$, such that the (on-shell) fermionic and bosonic degrees of freedom in the multiplet are both equal to 2. In $\mathcal{N}=2$ and $\mathcal{N}=4$, there are 2 respectively 4 chiral spinors, denoted by $\Omega^i(x)$ and $\psi^j(x)$ (here the position of the index indicates the chirality). The latter have to be supplemented by additional scalars to obtain the correct number of bosonic degrees of freedom. In $\mathcal{N}=2$ we have 1 complex scalar field $z(x)$ and in $\mathcal{N}=4$ there are 6 real scalars $\phi_{jk}(x)=-\phi_{kj}(x)$.

Besides the vector multiplets, there is also the chiral multiplet in $\mathcal{N}=1$, which consists of a complex scalar and a chiral fermion, and a hypermultiplet in $\mathcal{N}=2$, which contains 4 scalars and 2 chiral fermions.

Remark that for $\mathcal{N}>4$, matter multiplets exist in principle, but there is no interacting field theory known for these multiplets. The reason is that they all contain fields with spin $\geq 3/2$ and hence require *local* supersymmetry. By this we mean that a theory with spin-3/2 fields should automatically be invariant under supersymmetry transformations with space-time dependent parameters. At this point we only consider global parameters. Since we will be dealing with $\mathcal{N}=1$ supersymmetric theories in chapter 5, we are mainly interested in the global transformations of the $\mathcal{N}=1$ chiral and vector multiplets. There is only one supersymmetry generator $Q_\alpha$, and the associated parameter will be denoted by $\epsilon_\alpha$, which is also a spinor. Infinitesimal supersymmetry transformations are defined as follows: $\delta(\epsilon)\equiv\bar{\epsilon}^\alpha Q_\alpha$.

The components of the on-shell chiral multiplet are given by a complex scalar $z(x)$ and a chiral spinor $\chi_{(L)}(x)$. It will be beneficial, though, to formulate the supersymmetry transformations without the use of field equations. Therefore, we need to introduce a complex auxiliary field, $h(x)$, that can later be eliminated through its equations of motion. Then the transformation rules of the chiral multiplet are

$$\begin{aligned}
\delta(\epsilon)z &= \bar{\epsilon}_{(L)}\chi_{(L)}\,,\\
\delta(\epsilon)\chi_{(L)} &= \frac{1}{2}\gamma^\mu\epsilon_{(R)}\partial_\mu z+\frac{1}{2}h\epsilon_{(L)}\,,\\
\delta(\epsilon)h &= \bar{\epsilon}_{(R)}\gamma^\mu\partial_\mu\chi_{(L)}\,.
\end{aligned}\tag{3.34}$$

An explicit check reveals that the supersymmetry algebra (3.33) is indeed realized on the fields of the chiral multiplet.[6] In terms of the infinitesimal transformations $\delta(\epsilon)$ the algebra has the form

$$[\delta(\epsilon_1),\delta(\epsilon_2)]=-\bar{\epsilon}_{1(L)}\gamma^\mu\epsilon_{2(R)}\partial_\mu\,.\tag{3.35}$$

[6]If we eliminate the auxiliary fields $h(x)$ by means of their field equations, the algebra only closes off-shell, i.e. moduli the field equations.

In the case of the vector multiplet, we also need an extra auxiliary field, denoted by $D(x)$. The supersymmetry transformations of the vector, spinor and auxiliary fields are

$$
\begin{aligned}
\delta(\epsilon)A_\mu &= -\frac{1}{2}\bar{\epsilon}_{(L)}\gamma_\mu\lambda_{(R)} - \frac{1}{2}\bar{\epsilon}_{(R)}\gamma_\mu\lambda_{(L)}\,, \\
\delta(\epsilon)\lambda_{(R)} &= \frac{1}{4}\left(\gamma_{\mu\nu}F^{\mu\nu} - 2\mathrm{i}D\right)\epsilon_{(R)}\,, \\
\delta(\epsilon)D &= \frac{\mathrm{i}}{2}\bar{\epsilon}_{(L)}\gamma^\mu\partial_\mu\lambda_{(R)} - \frac{\mathrm{i}}{2}\bar{\epsilon}_{(R)}\gamma^\mu\partial_\mu\lambda_{(L)}\,,
\end{aligned}
\tag{3.36}
$$

where the field strength $F^{\mu\nu}$ is defined as in (2.8). Also for these transformations, the supersymmetry algebra (3.35) is satisfied.

In order to introduce the dynamics of the fields, one needs to construct an action that is invariant under the transformations (3.34) and (3.36). There exists a general formalism, known as multiplet calculus [8, 32–35], that has exactly this function. It starts from the observation that the components $h(x)$ and $D(x)$ transform into a total derivative under supersymmetry. Therefore the actions

$$
S_F = \int \mathrm{d}^4x\, h(x)\,, \qquad S_D = \int \mathrm{d}^4x\, D(x) \tag{3.37}
$$

are supersymmetric. The next step is to choose $h$ and $D$ as the components of a composite multiplet that has been constructed from a combination of basic chiral and vector multiplets. The results of this procedure will be briefly reviewed in the next section.

### Supersymmetric actions

Consider $n_C$ chiral multiplets with components $(z^i, \chi^i_{(L)}, D^i)$, $i = 1, \ldots, n_C$. If we perform the correct steps in the multiplet calculus starting from an arbitrary real function $\mathcal{K}(z^i, \bar{z}^{\bar{\imath}})$ of the scalars $z^i$ and their complex conjugates $\bar{z}^{\bar{\imath}}$, we obtain the following supersymmetric action for the chiral multiplets:

$$
S_{\text{chiral}} = -\int \mathrm{d}^4x\; g_{i\bar{\jmath}}\left(\partial_\mu z^i\partial^\mu\bar{z}^{\bar{\jmath}} + \bar{\chi}^i_{(L)}\gamma^\mu\partial_\mu\chi^{\bar{\jmath}}_{(R)} + \bar{\chi}^{\bar{\jmath}}_{(R)}\gamma^\mu\partial_\mu\chi^i_{(L)} + h^i\bar{h}^{\bar{\jmath}}\right) + \ldots\,. \tag{3.38}
$$

The first term is the kinetic action of the scalar fields. It takes the form of a non-linear sigma model, with the scalars $(z^i, \bar{z}^{\bar{\imath}})$ as coordinates on the target space with metric $g_{i\bar{\jmath}}(z^i, \bar{z}^{\bar{\imath}})$. The latter is given by the second derivative of the function $\mathcal{K}(z^i, \bar{z}^{\bar{\imath}})$, which can therefore be identified with the Kähler potential. More details on the scalar geometry, and its connection to the global symmetries of the theory, will be given in §4.1.

The second and third term in (3.38) correspond to the kinetic terms of the chiral fermions. Finally, the dots in (3.38) suggest the presence of other terms, which are interaction terms between the scalars and fermions, as well as 4-fermion terms and a superpotential. Except for the contributions from the superpotential, which are supersymmetric on their own, the other terms are uniquely fixed in a supersymmetric action. We have not written them down since they will not be important for the remainder of our story.

Similar to the construction of $S_{\text{chiral}}$, the multiplet calculus can also be used to determine the action for the vector multiplets. Consider $n_V$ Abelian multiplets with components $({A_\mu}^\Lambda, \lambda^\Lambda_{(R)}, D^\Lambda)$, $\Lambda = 1, \ldots, n_V$. The index $\Lambda$ can be thought of as a gauge index, enumerating the different Abelian generators in the gauge group $G_{\text{gauge}} = U(1)^{n_V}$. Remark that for the moment, we confine our discussion to the direct product of $n_V$ Abelian $U(1)$ factors. The generalization to arbitrary gauge groups, which opens the perspective of constructing more realistic supersymmetric field theories, will be discussed in §4.2.

In order to obtain the action for the Abelian vector multiplets, we start from the lowest component of a composite chiral multiplet, $z(fW^2) = -\frac{1}{2} f_{\Lambda\Sigma} \bar\lambda^\Lambda_{(L)} \lambda^\Sigma_{(L)}$, with $f_{\Lambda\Sigma}(z^i) = f_{\Sigma\Lambda}(z^i)$ a holomorphic function of the scalar fields $z^i$. We could also have taken $f_{\Lambda\Sigma} = \delta_{\Lambda\Sigma}$, but in order to be fully general, we will include the interactions with the scalar multiplets via a non-trivial function $f_{\Lambda\Sigma}$. Using the multiplet calculus, one obtains the highest component $h(fW^2)$ corresponding to $z(fW^2)$. According to (3.37), this is the integrand of a supersymmetric action:

$$\begin{aligned} S_{\text{vector}} \;=\; \int \mathrm{d}^4x \Big( & -\frac{1}{4} \operatorname{Re} f_{\Lambda\Sigma} {F_{\mu\nu}}^\Lambda F^{\mu\nu\Sigma} - \frac{1}{2} \operatorname{Re} f_{\Lambda\Sigma} \bar\lambda^\Lambda \gamma^\mu \partial_\mu \lambda^\Sigma \\ & + \frac{1}{8} \operatorname{Im} f_{\Lambda\Sigma} \varepsilon^{\mu\nu\rho\sigma} {F_{\mu\nu}}^\Lambda F^\Sigma_{\rho\sigma} + \frac{1}{4} \mathrm{i} (\partial_\mu \operatorname{Im} f_{\Lambda\Sigma}) \bar\lambda^\Lambda \gamma^5 \gamma^\mu \lambda^\Sigma \\ & + \frac{1}{2} \operatorname{Re} f_{\Lambda\Sigma} D^\Lambda D^\Sigma \Big) + \ldots \end{aligned} \tag{3.39}$$

The first term is the usual kinetic term for the vector fields. In particular, if we choose $f_{\Lambda\Sigma} = \delta_{\Lambda\Sigma}$ it reduces to the Maxwell action in (2.11). The second term is the kinetic action for the gauginos, which is also proportional to the real part of the gauge kinetic function $f_{\Lambda\Sigma}$. In the second line we find two contributions that are characteristic for supersymmetric theories. The first term is named after Peccei and Quinn, who first proposed its existence (although in a different context) to solve the strong CP problem. But in contrast to (2.63), it is not a total derivative because $\operatorname{Im} f_{\Lambda\Sigma}$ is scalar field-dependent. The second term on the second line reveals the chiral nature of $\mathcal{N} = 1$ supersymmetric theories; it contains the chiral coupling (i.e., involving a $\gamma_5$-matrix) of the gauginos. Finally, the dots denote again extra terms that describe the interactions between the vectors and spinors.

Although we have restricted our attention to the $\mathcal{N} = 1$ multiplets from Table 3.1, the form of the actions (3.38) and (3.39) is remarkably generic. For all amounts of supersymmetry in four dimensions, the bosonic part of the kinetic action has the form

$$\begin{aligned} S_{\text{kin, bos}} &= \int \mathrm{d}^4x \left( -G_{ij}(\phi)\partial_\mu \phi^i \partial^\mu \phi^j + \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}(\phi) F_{\mu\nu}{}^{\Lambda} F^{\mu\nu\Sigma} \right. \\ &\qquad \left. - \frac{1}{8}\mathcal{R}_{\Lambda\Sigma}(\phi)\varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu\Lambda} F^{\rho\sigma\Sigma} \right) . \end{aligned} \tag{3.40}$$

The scalars $\phi^i$ always appear in the form of a non-linear sigma model. They can be seen as the coordinates on some geometrical space endowed with a metric $G_{ij}(\phi)$. The properties of this space are determined by the amount of supersymmetry.[7] Furthermore, $\mathcal{R}_{\Lambda\Sigma}$ and $\mathcal{I}_{\Lambda\Sigma}$ are the real and imaginary parts of some scalar dependent function $\mathcal{N}_{\Lambda\Sigma}(\phi) = \mathcal{R}_{\Lambda\Sigma}(\phi) + \mathrm{i}\mathcal{I}_{\Lambda\Sigma}(\phi)$. The function $\mathcal{N}_{\Lambda\Sigma}(\phi)$ has to satisfy certain conditions, also depending on the amount of supersymmetry.

The matter actions we have constructed so far live on a flat Minkowski space and are invariant under global Poincaré and supersymmetry transformations. In the next section we will show how they can be consistently coupled to gravity, leading to supergravity theories that are invariant under local diffeomorphisms and local supersymmetry transformations.

## 3.3 $D = 4$ supergravity

In a nutshell, supergravity can be described as a “supersymmetric theory of gravity”. There are basically two equivalent ways of looking at such theories. The first way assumes the validity of general relativity as a description of gravity, and one proceeds by adding additional matter in a way that is compatible with supersymmetry. However, from a constructive point of view, this is not an interesting perspective. Therefore, we will reverse the argument and introduce supergravity starting from a globally supersymmetric theory and applying the gauging procedure. This second method is analogous to §3.1 where Einstein gravity emerged as the gauge theory for Poincaré transformations. For clarity and in view of our needs in chapter 5, we will work exclusively in 4 dimensions with an emphasis on $\mathcal{N} = 1$ theories.

[7] In $\mathcal{N} = 1$, for example, we remember from our discussion below (3.38) that it needs to be Kähler.

### Basic supergravity ingredients

Recall that the invariance of a theory under *local translations* is equivalent to demanding diffeomorphism invariance and hence the presence of gravity. The same procedure also works for supergravity, but instead of gauging the Poincaré algebra, we will demand the invariance under *local supersymmetry transformations.* It means that the global supersymmetry parameters $\epsilon^i_\alpha$ become space-time dependent functions $\epsilon^i_\alpha(x)$. If one combines this feature with the form of the supersymmetry algebra,

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = -\bar{\epsilon}^i_1 \gamma^\mu \epsilon_{2i} \partial_\mu \, , \tag{3.41}$$

one expects to find local translations with parameters $\xi^\mu(x) = -\bar{\epsilon}^i_1(x)\gamma^\mu\epsilon_{2i}(x)$. This is how gravity enters into the game.

The invariance of a theory under local supersymmetry can only be achieved after the introduction of an appropriate set of gauge fields. These fields constitute the so-called gauge (or supergravity) multiplet of a supergravity theory. Besides the original vierbein $e_\mu{}^a(x)$ and the auxiliary spin connection $\omega_\mu{}^{ab}(x)$, it also contains the gauge fields associated to the supersymmetry generators $Q^i_\alpha$. We will denote them by $\Psi^i_{\mu\alpha}(x)$, and since they are the superpartners of the graviton $e_\mu{}^a(x)$, we will refer to them as "gravitini". Note that the amount of gravitini, and hence the precise field content of the supergravity multiplet, depends on the number of supercharges, $\mathcal{N}$.

In general, we also have to include other fields such as the vectors $A_{\mu A}(x)$ that correspond to the $R$-symmetry generators $T_A$. Part of these fields might be auxiliary fields, such as the $R$-symmetry gauge field in $\mathcal{N} = 1$. In the remainder of this discussion, all auxiliary fields will be left out. In the end, one obtains the on-shell field content in Table 3.2 for the various 4-dimensional supergravity multiplets.

**Table 3.2:** Number of fields corresponding to each spin $s$-value. Only the supergravity multiplets for $\mathcal{N} = 1, 2$ and 8 are listed.

| | $s=2$ | $s=3/2$ | $s=1$ | $s=1/2$ | $s=0$ |
|---|---|---|---|---|---|
| $\mathcal{N}=1$ | 1 | 1 | | | |
| $\mathcal{N}=2$ | 1 | 2 | 1 | | |
| $\vdots$ | | | | | |
| $\mathcal{N}=8$ | 1 | 8 | 28 | 56 | 70 |

For $\mathcal{N} = 1$ we have one spin-1 field, which is the graviton, and one spin-3/2 field, which is the gravitino corresponding to the single supersymmetry operator $Q_\alpha$. For higher values of $\mathcal{N}$ the field content becomes increasingly more complicated.

The reader should notice that the values of $\mathcal{N}$ are not restricted to 4 anymore, since gravity *requires* spin-3/2 and spin-2 fields (remember our discussion below Table 3.1). However, it does also not exceed the value 8, since for $\mathcal{N} > 8$, the multiplets necessarily contain fields with spin $\geq 5/2$, which do not have a consistent dynamical description within the framework of an interacting field theory.

Let us now postulate the local supersymmetry transformations of the different gauge fields, in particular for the graviton and gravitini. The most general transformation of the vierbein that is consistent with its index structure is given by

$$\delta(\epsilon) e_\mu{}^a = \frac{1}{2} \bar{\epsilon}^i(x) \gamma^a \Psi_\mu^i(x) \,, \tag{3.42}$$

where a sum over the $i$-index is understood. The transformations of the gravitini are

$$\delta(\epsilon) \Psi^i_{\mu\alpha} = \partial_\mu \epsilon^i_\alpha(x) + \frac{1}{4} \omega_\mu{}^{ab} (\gamma_{ab})_\alpha{}^\beta \epsilon^i_\beta(x) + \dots \,. \tag{3.43}$$

The first term is as expected. The second term looks like a local Lorentz transformation, with the parameter replaced by the gauge field $\omega_\mu{}^{ab}$. In contrast to §3.1, the spin connection $\omega_\mu{}^{ab}$ is now the one with torsion, as we discussed in footnote 3 of that section. The contortion tensor $T_\mu{}^{ab}$ is bilinear in the gravitini. Finally, the dots in (3.43) denote extra contributions that depend on the other fields in the supergravity multiplet.

Given these transformations, one can verify the local supersymmetry algebra, which is a modified version of (3.33). If one includes the auxiliary fields in the supergravity multiplet, the commutator of two local supersymmetry transformations now closes into general coordinate transformations (gct), but also local Lorentz transformations (ll) and local supersymmetry transformations (susy). For example,

$$[\delta(\epsilon_1), \delta(\epsilon_2)] e_\mu{}^a = D_\mu \xi^a = [\delta_{\rm gct}(\xi) - \delta_{\rm ll}(\lambda_3) - \delta_{\rm susy}(\epsilon_3)]\, e_\mu{}^a \,, \tag{3.44}$$

with $\xi^\mu$, $\lambda_3^{ab}$ and $\epsilon_3^i$ new field-dependent parameters.

### Pure supergravity action

We are now ready to present the supersymmetric extension of the Einstein-Hilbert action (3.27). This extension depends on all the fields in the supergravity multiplet and it should be invariant under the transformations (3.42), (3.43), etc. For simplicity we will only state its $\mathcal{N} = 1$ version:

$$S_{\rm sugra} = \frac{1}{2} \int {\rm d}^4 x \, e \left( e_a{}^\mu e_b{}^\nu R_{\mu\nu}{}^{ab}(\omega) - \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \hat{\partial}_\nu \Psi_\rho \right) \,, \tag{3.45}$$

where $\hat{\partial}_\nu \Psi_\rho \equiv \partial_\nu \Psi_\rho + \frac{1}{4} \omega_\nu{}^{ab} (\gamma_{ab}) \Psi_\rho$ is a supercovariant derivative. The first term in $S_{\rm sugra}$ is equal to the usual Einstein-Hilbert action (3.27), but the spin connection now contains a torsion term that is bilinear in the gravitini. The second term is called a Rarita-Schwinger term and forms the supersymmetric completion of the Einstein-Hilbert part of the action. In the end, an explicit calculation reveals that $S_{\text{sugra}}$ is indeed invariant under the transformations (3.42) and (3.43), as well as under general coordinate transformations and local Lorentz transformations.

### Matter coupled supergravity

The last section of this chapter is devoted to the interesting problem of coupling the matter multiplets from Table 3.1 to pure supergravity. This topic is a generalization of our original discussion in the third paragraph of §3.1 about the coupling of matter to basic Einstein gravity. First we need to change all ordinary space-time derivatives in (3.38) and (3.39) to covariant derivatives. Of course, this time "covariant" means "with respect to general coordinate transformations, local Lorentz transformations ánd local supersymmetry transformations".[8] Similar to (3.22), we also need to multiply the resulting Lagrangian with a factor $e$, which is the determinant of the vierbein. To finish the analogy, we have to add the action for the free gauge fields, i.e. $S_{\text{sugra}}$ in (3.45). In the case of $\mathcal{N}=1$ supergravity, this leads to the following result for the bosonic part of the kinetic action:

$$\begin{aligned} S_{\text{kin, bos}} \quad = \quad & \int \mathrm{d}^4 x\, e \left( \frac{1}{2} R(e) - g_{i\bar{\jmath}} \hat{\partial}_\mu z^i \hat{\partial}^\mu z^{\bar{\jmath}} \right. \\ & \left. - \frac{1}{4} \mathrm{Re}\, f_{\Lambda\Sigma} F_{\mu\nu}{}^\Lambda F^{\mu\nu\Sigma} + \frac{1}{8} \mathrm{Im}\, f_{\Lambda\Sigma} e^{-1} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}{}^\Lambda F_{\rho\sigma}{}^\Sigma \right), \end{aligned} \tag{3.46}$$

where $\hat{\partial}_\mu$ is used to indicate a supercovariant derivative. The fermionic part of the kinetic Lagrangian has the form

$$\begin{aligned} S_{\text{kin, ferm}} \quad = \quad & \int \mathrm{d}^4 x\, e \left( -\frac{1}{2} \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \hat{\partial}_\nu \Psi_\rho \right. \\ & - g_{i\bar{\jmath}} \left[ \bar{\chi}^i_{(L)} \gamma^\mu \hat{\partial}_\mu \chi^{\bar{\jmath}}_{(R)} + \bar{\chi}^{\bar{\jmath}}_{(R)} \gamma^\mu \hat{\partial}_\mu \chi^i_{(L)} \right] \\ & \left. - \frac{1}{2}\, \mathrm{Re}\, f_{\Lambda\Sigma} \bar{\lambda}^\Lambda \gamma^\mu \hat{\partial}_\mu \lambda^\Sigma + \frac{1}{4} \mathrm{i} (\hat{\partial}_\mu\, \mathrm{Im}\, f_{\Lambda\Sigma}) \bar{\lambda}^\Lambda \gamma^5 \gamma^\mu \lambda^\Sigma \right). \end{aligned} \tag{3.47}$$

Again the hatted derivatives are fully covariant with respect to all local super-Poincaré transformations.

[8]We will see in equation (3.53) that the derivatives are also covariant with respect to Kähler transformations. These are local transformations that correspond to the $U(1)$ $R$-symmetry of the $\mathcal{N}=1$ superalgebra. Recall that the corresponding gauge field $\mathcal{A}_\mu$ is an auxiliary field in the supergravity multiplet, and can therefore be integrated out using its equations of motion.

However, this is not yet the final result because $S_{\text{kin, bos}}$ and $S_{\text{kin, ferm}}$ are not invariant under transformations (3.34) and (3.36) where all supersymmetry parameters are replaced by local functions $\epsilon = \epsilon(x)$. In order to restore local supersymmetry invariance, one needs to add an intricate set of corrections to the actions and the field transformations. Several systematic methods exist that can be used to obtain these corrections. The first method makes use of the superfield formalism [27]. This approach is especially useful in minimal supergravity, but becomes very cumbersome for higher amounts of supersymmetry. An alternative method relies on the straightforward gauging of supersymmetry. One determines by hand the different corrections that are needed to the action and the transformation laws, such that the action is invariant under these transformations. This is called the Noether method, and is very convenient for theories with only a few fields, such as $11D$ supergravity [36]. The third and most powerful method is superconformal tensor calculus [32, 33, 35, 37–39]. It is an extension of the ordinary multiplet calculus, which we have briefly mentioned in §3.2 as a useful tool for the construction of globally supersymmetric actions. It has been successfully applied to $\mathcal{N} = 1, 2, 4$ theories in 4, 5 and 6 spacetime dimensions [9, 35, 40–44].

At the moment we will not pursue a detailed analysis of any of these methods, since the most relevant terms for our discussion are already contained in $S_{\text{kin, bos}}$ and $S_{\text{kin, ferm}}$.[9] It suffices to say that in general, extra contributions to the action consist of 4-fermion terms, a potential $V(z, \bar{z})$ for the scalar fields and bilinear mass terms for the fermions. The only transformation laws that get corrections are those of the fermions. They pick up extra terms that depend on the scalar; for the gaugino's and the chiralino's these are a consequence of integrating out the auxiliary fields $h^i(x)$ and $D^\Lambda(x)$. Moreover, there are cubic terms, i.e. quadratic in the fermion fields multiplied by $\epsilon$:

$$
\begin{aligned}
\delta\Psi_{\mu(L)} &= \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} - \frac{1}{2}\mathrm{i}\mathcal{A}_\mu\right)\epsilon_{(L)} + \frac{1}{2}\gamma_\mu \mathrm{e}^{\mathcal{K}/2} W \epsilon_{(R)} + \text{cubic}\,, \\
\delta\chi^i_{(L)} &= \frac{1}{2}\gamma^\mu\partial_\mu z^i \epsilon_{(R)} - \frac{1}{2}\mathrm{e}^{\mathcal{K}/2} g^{i\bar{\jmath}}\left(\partial_{\bar{\jmath}} + \partial_{\bar{\jmath}}\mathcal{K}\right)\overline{W}\epsilon_{(L)} + \text{cubic}\,, \\
\delta\lambda^\Lambda &= \left(\frac{1}{4}\gamma^{\mu\nu}F_{\mu\nu}{}^\Lambda + \frac{1}{2}\mathrm{i}\gamma_5(\mathrm{Re}\, f)^{-1\Lambda\Sigma}\mathcal{P}_\Sigma\right)\epsilon + \text{ cubic}\,.
\end{aligned}
\tag{3.48}
$$

Given these (partial) results about the basic matter-coupled supergravity action and the corresponding field transformations, the most important conclusion we can draw is that their structure is essentially fixed by diffeomorphism invariance

[9] A more careful analysis of the $\mathcal{N} = 1$ superfield approach will be required in §5.3, though, and we refer the interested reader to that section for more details. Also some aspects of $\mathcal{N} = 1$ superconformal tensor calculus will be used in chapter 5, but we refer to [8, 9] for a complete analysis.

and local supersymmetry requirements. However, it is also clear that some input still remains to be specified, namely the scalar and vector kinetic matrices $g_{i\bar{\jmath}}$ and $f_{\Lambda\Sigma}$, as well as the superpotential $W$ (if present). These objects are again not entirely arbitrary, but constrained by supersymmetry. Let us give some more details about their form and the constraints they need to satisfy.

(i) The *gauge kinetic function* $f_{\Lambda\Sigma}(z)$. In $\mathcal{N}=1$ global and local supersymmetry, $f_{\Lambda\Sigma}(z)$ simply has to be holomorphic in the complex scalars of the chiral multiplets.

(ii) The *superpotential* $W(z)$. Also this object has to be a holomorphic function of the complex scalars. The superpotential appears quadratically in the $F$-term of scalar potential $V(z,\bar{z})$, and linearly in the fermionic mass terms.

(iii) The *Kähler potential* $\mathcal{K}(z,\bar{z})$. As we saw before, it fixes the form of the metric $g_{i\bar{\jmath}}$ on the scalar manifold, up to Kähler transformations

$$\mathcal{K}(z,\bar{z}) \;\rightarrow\; \mathcal{K}(z,\bar{z}) + F(z) + \bar{F}(\bar{z})\,. \tag{3.49}$$

Moreover, one must verify that the total action is invariant under these transformations if it is to be well-defined over the entire Kähler manifold. One can show that the transformations (3.49) must be accompanied by chiral rotations of the fermions and an appropriate transformation of the superpotential;

$$\begin{aligned}
&\Psi_\mu \rightarrow \mathrm{e}^{-\mathrm{i}(\operatorname{Im} F)\gamma_5/2}\Psi_\mu, \quad \lambda^\Sigma \rightarrow \mathrm{e}^{-\mathrm{i}(\operatorname{Im} F)\gamma_5/2}\lambda^\Sigma, \quad \chi^i \rightarrow \mathrm{e}^{\mathrm{i}(\operatorname{Im} F)\gamma_5/2}\chi^i,\\
&W \rightarrow \mathrm{e}^{-F}W\,.
\end{aligned} \tag{3.50}$$

The bosonic fields, on the other hand, all have Kähler weight zero and do not transform.[10] In fact, the Kähler transformations are *local* $U(1)$ symmetries of the theory with gauge parameters $\theta(x) \equiv \operatorname{Im} F(z)$. The corresponding $U(1)$ gauge field, denoted by $\mathcal{A}_\mu$, is a composite field that depends on the scalars[11] in the theory,

$$\mathcal{A}_\mu = \frac{1}{2}\mathrm{i}\left(\partial_\mu z^i \partial_i \mathcal{K} - \partial_\mu z^{\bar{\imath}} \partial_{\bar{\imath}} \mathcal{K}\right)\,. \tag{3.51}$$

One can easily show that $\mathcal{A}_\mu$ transforms with the derivative of the gauge parameter under Kähler transformations,[12]

$$\delta_{\mathrm{K}}\mathcal{A}_\mu = -\partial_\mu(\operatorname{Im} F) = -\partial_\mu\theta(x)\,, \tag{3.52}$$

[10] Remark that, in spite of the invariance of the scalars under Kähler transformations, the scalar-dependent functions $\mathcal{K}(z,\bar{z})$ and $W(z)$ do transform!

[11] We assume the absence of Fayet-Iliopoulos constants for a moment.

[12] We will add a subscript K to the $\delta$-symbol to indicate that we are calculating a Kähler transformation.

Moreover, all covariant derivatives $\hat{\partial}_\mu$ that appear in the Lagrangians (3.46) and (3.47) are covariant with respect to the $U(1)$ Kähler transformations:

$$\hat{\partial}_\mu \psi_\mu = \ldots - \frac{1}{2}\mathrm{i}\mathcal{A}_\mu \gamma_5 \psi_\mu, \qquad \hat{\partial}_\mu \lambda^\Sigma = \ldots - \frac{1}{2}\mathrm{i}\mathcal{A}_\mu \gamma_5 \lambda^\Sigma,$$

$$\hat{\partial}_\mu \chi^i_{(L)} = \ldots + \frac{1}{2}\mathrm{i}\mathcal{A}_\mu \chi^i_{(L)} . \tag{3.53}$$

Once we have made a choice for the vector kinetic matrix $f_{\Lambda\Sigma}(z)$, the superpotential $W(z)$, and the Kähler potential $\mathcal{K}(z, \bar{z})$, the action and corresponding field transformations are completely fixed.

## 3.4 Summary and outlook

This concludes our discussion about the general structure of supergravity theories. Their construction is mainly based on symmetry-principles, in particular on the requirement of local super-Poincaré invariance. These symmetries highly constrain the field content of the theory –which is organized in a gravity multiplet and additional matter multiplets– and the couplings in the Lagrangian. We have tried to give some intuition about the different steps in the construction of such invariant Lagrangians and the appropriate transformations of the fields.

As a result, we found that supergravity theories exist in many varieties, despite their universal form. Their properties depend primarily on the choice of spacetime dimension $D$ (not explicitly discussed here) and the number of supercharges $\mathcal{N}$. For some values of $D$ and $\mathcal{N}$ one also has the freedom to decide which type of matter multiplets can be added. For example, for $D = 4$ the possibilities were given in Table 3.1. Over the years, the various supergravities –also called basic supergravities– have been classified, and they are well understood.

However, when the field content is fixed after the above steps, we saw that there is still some freedom left. The simplest example being the choice of the Kähler and superpotential in $D = 4$, $\mathcal{N} = 1$ supergravity, as we pointed out in §3.3. Other types of "deformations" consist in the addition of mass parameters [45] or the coupling to Yang-Mills type gauge groups [46]. The last two types of deformations have not been discussed in this chapter, but they will be our main point of interest in the remainder of this text. In the next chapter, we will commence our discussion about the reconciliation of Yang-Mills gauge structures with basic supergravity structures. Put differently, we will see how chapters 2 and 3 of this thesis can be merged into one framework, called "gauged supergravity".

CHAPTER 4

# GAUGED SUPERGRAVITY

Basic supergravity theories in $D$ dimensions with a fixed number of supercharges and a well-defined matter content are not uniquely determined. For instance, in the previous chapter we saw that $D = 4$, $\mathcal{N} = 1$ theories with chiral and vector multiplets depend on three variable functions: the vector kinetic function, the Kähler potential and the superpotential. Depending on the desired properties of the theory, an appropriate choice for these functions must be made.

In addition to the scalar functions, there exist other types of "deformations" that influence the final form of the supergravity Lagrangian and the local field transformations. The most familiar set of examples are the so-called "gauged supergravities". These were first analyzed in the early 80's, when $4D$ basic supergravity with a maximal number of supercharges was reconciled with the non-Abelian gauge structure of Yang-Mills theories [47]. The starting point of this construction is the existence of a large set of global internal symmetries that transform the fields but leave the theory itself invariant.[1] The Yang-Mills-type deformations can then be found if one promotes these global symmetries (or part of them) to *local invariances* of the theory. This corresponds to a gauging in the same way as we discussed in chapter 2, besides some non-trivial modifications that guarantee the consistency with supersymmetry.

Apart from the gauging of theories with a maximal amount of supersymmetry,

[1] For $4D$ extended supergravity, these symmetries are generally non-compact [48–56].

such deformations generically exist for most supergravities irrespective of the amount of supersymmetry or the space-time dimension. In each case, there exist global internal symmetries that form a group, denoted by $G_{\text{global}}$. Some symmetries of this group are realized at the level of the action, whereas others are only demonstrated to leave the equations of motion invariant. The former can be promoted to local invariances of the theory, thereby giving rise to a gauged supergravity. Our goal in this chapter is to learn how to construct the most general possible actions for these gauge theories. As we pointed out in the introduction, these actions can then be used by model builders in their search for realistic theories.

The organization of this chapter is as follows. Section 4.1 contains a detailed review of the global internal symmetries of $4D$ supergravities. In particular, the properties of $G_{\text{global}}$ and its representation on the fields will be determined for theories with a minimal amount of supersymmetry (although the main points can easily be extended to any $4D$ theory). In general, we find that vectors and scalars transform in linear and non-linear representations of $G_{\text{global}}$, respectively, whereas the space-time metric $g_{\mu\nu}$ is invariant. In §4.2 we will use these results and discuss how part of the global symmetry group can be promoted to local symmetries via the introduction of minimal couplings to the gauge fields. This requires the implementation of our methods and structures from chapter 2. Finally, in §4.3, a generalization of this conventional approach will be presented in the form of the embedding tensor mechanism. This formalism provides a natural framework for the systematic construction and classification of *all* possible gaugings. We will review its basic structure and present the aspects that are most useful for chapters 6 and 7 of this thesis.

## 4.1 Symmetries and dualities

The determination of the global internal symmetries of a basic supergravity theory is the first step towards the construction of its gauged deformations. The basic theories have been discussed in the previous chapter and for the purposes of this section, it suffices to focus on the bosonic part of their kinetic action (3.46).[2] Depending on the couplings in the theory, it is possible to have symmetry transformations that act only on certain fields and not on the rest, and symmetries that act simultaneously on all of them. This adds a great deal of complication to the final result. In order to make our discussion as transparent as possible, we will divide it into two parts. The first part, §4.1, is concerned with the symmetry transformations of the scalar fields, whereas the second part, §4.1, deals with

[2]Obviously one also has to take the fermions into account if one wants to discuss symmetries of the full theory. However we will see that an extension of our discussion to the fermion fields is rather straightforward.

the fields in the vector multiplet. Both parts can be treated independently to a great extent, although we will find an intricate relationship via the gauge kinetic functions $\mathcal{N}_{\Lambda\Sigma}(\phi)$ that couple the scalars to the vector fields. Since this section contains a detailed discussion with a lot of new notations, we will summarize our results in §4.1 in a way that is intended to be both concise and useful.

### Scalar sector

As we saw in §3.3, supergravity theories generically contain scalar fields $\phi^i$, whose kinetic terms take the form of a non-linear sigma model:

$$\mathcal{L}_{\text{kin, scalar}} = -\frac{1}{2}\, e\, G_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j\,. \tag{4.1}$$

As a consequence, the scalars can be interpreted as the coordinates on a target space that is equipped with a metric $G_{ij}(\phi)$. Supersymmetry places strong constraints on the possibilities for these spaces, see e.g. [57, 58]. In the case of $\mathcal{N}=1$ supergravity, the target space is a Hodge Kähler manifold[3] and the metric can be written as the second derivative of the Kähler potential,

$$g_{i\bar{\jmath}}(z,\bar{z}) = \partial_i\partial_{\bar{\jmath}}\mathcal{K}(z,\bar{z})\,. \tag{4.2}$$

For general $\mathcal{N}$, the scalar geometries are listed in Table 4.1. One sees that theories

**Table 4.1:** Scalar geometries of $4D$ supergravity theories.

| $\mathcal{N}$ | target space geometry |
|---|---|
| 1 | Hodge Kähler |
| 2 | special Kähler × quaternionic Kähler |
| 4 | $\frac{\mathrm{SL}(2)}{\mathrm{SO}(2)} \times \frac{SO(6,6+n_V)}{SO(6)\times SO(6+n_V)}$ |
| 8 | $E_{7(+7)}/SU(8)$ |

with more supersymmetry have stronger constraints on their target space. For $\mathcal{N}=8$ maximal supergravity, for example, this space is fixed to be a $E_{7(+7)}/SU(8)$ coset. It suffices to say that theories with a $G/K$ symmetric coset space can be studied using the techniques of symmetric spaces. For more information about symmetric spaces we refer to [59].

In the remainder of this section we will concentrate on the properties of $\mathcal{N}=1$ supergravity. Let us determine the scalar transformations that preserve the

[3] Since the supergravity action is not necessarily well defined over a general Kähler manifold, one needs to impose an extra restriction which is the Hodge condition. For global supersymmetric theories, such a restriction is not necessary.

complex structure of the Kähler manifold and that leave the kinetic term (4.1) invariant. The first requirement is satisfied if we restrict to holomorphic coordinate transformations

$$\delta(\Lambda)z^i = \Lambda^\alpha k_\alpha^i(z)\,, \quad \delta(\Lambda)z^{\bar\imath} = \Lambda^\alpha k_\alpha^{\bar\imath}(\bar z)\,, \tag{4.3}$$
$$\text{with} \qquad \partial_{\bar\jmath} k_\alpha^i(z) = 0 \quad \text{and} \quad \partial_j k_\alpha^{\bar\imath}(\bar z) = 0\,.$$

The $\alpha$-index labels all independent such transformations, and the corresponding constant parameters are denoted by $\Lambda^\alpha$. The $k_\alpha^i(z)$ and $k_\alpha^{\bar\imath}(\bar z)$ that generate the transformations are arbitrary holomorphic and antiholomorphic functions respectively. They close into an algebra with structure constants $f_{\alpha\beta}{}^\gamma$ if the following condition is satisfied[4]

$$k_\alpha^i \partial_i k_\beta^j - k_\beta^i \partial_i k_\alpha^j = f_{\alpha\beta}{}^\gamma k_\gamma^i\,. \tag{4.4}$$

The group that corresponds to this algebra will be denoted by $G_{\mathrm{scalar}}$.

Once we require invariance of the kinetic term (4.1) under the transformations (4.3), we have to impose another constraint on the functions $k_\alpha^i(z)$ and $k_\alpha^{\bar\imath}(\bar z)$:

$$g_{k\bar\jmath}\partial_i k_\alpha^k + g_{i\bar k}\partial_{\bar\jmath} k_\alpha^{\bar k} + \left(k_\alpha^k \partial_k + k_\alpha^{\bar k}\partial_{\bar k}\right) g_{i\bar\jmath} = 0\,. \tag{4.5}$$

This requirement is equivalent to the Killing equation $\mathcal{L}_k g_{i\bar\jmath} = 0$, and the symmetries (4.3) are therefore given by the (holomorphic) isometries of the Kähler manifold, i.e., $G_{\mathrm{scalar}} = \mathrm{Iso}(\mathcal{M}_{\mathrm{scalar}})$. Note that locally, the Killing equation implies the existence of real scalar functions $\mathcal{P}_\alpha(z,\bar z)$ such that

$$\mathrm{i} g_{i\bar\jmath} k_\alpha^{\bar\jmath} = \partial_i \mathcal{P}_\alpha\,. \tag{4.6}$$

The functions $\mathcal{P}_\alpha$ are called the Killing potentials or moment maps. They are only defined up to real integration constants which will be denoted by $\xi_\alpha$. The latter can be related to the Fayet-Iliopoulos constants of the theory.

Given the necessary and sufficient condition (4.5) on the scalar transformations, such that the kinetic term (4.1) is invariant, one can now proceed to show invariance of the full theory. In particular, one needs to determine the transformations of the Kähler potential, the superpotential and the gauge kinetic functions. We start with a short calculation that leads to the following transformation of the Kähler potential:

$$\delta(\Lambda)\mathcal{K} = \Lambda^\alpha \left(k_\alpha^i \partial_i \mathcal{K} + k_\alpha^{\bar\imath}\partial_{\bar\imath}\mathcal{K}\right)\,. \tag{4.7}$$

In general, the $\mathcal{K}$-dependent terms in the supergravity action will not be invariant under these transformations. However, we know that the action is invariant under

[4] The antiholomorphic functions satisfy the same algebra.

Kähler transformations $\mathcal{K}' = \mathcal{K} + F + \bar{F}$ (recall (3.49)) and therefore, we will require that (4.7) is proportional to such a transformation. One can check that this is only consistent with the Killing equation in (4.5) if

$$F = \Lambda^\alpha \left( \mathrm{i}\mathcal{P}_\alpha + k_\alpha^i \partial_i \mathcal{K} \right) . \tag{4.8}$$

Simultaneously, we also require a convenient transformation of the holomorphic superpotential,

$$\delta(\Lambda) W = \Lambda^\alpha k_\alpha^i \partial_i W = -FW . \tag{4.9}$$

and a suitable Kähler transformation of the fermions,

$$\delta\Psi_\mu = -\frac{\mathrm{i}}{2}(\mathrm{Im}\, F)\gamma_5 \Psi_\mu, \quad \delta\lambda^\Sigma = -\frac{\mathrm{i}}{2}(\mathrm{Im}\, F)\gamma_5 \Lambda^\Sigma, \quad \delta\chi^i_{(L)} = \frac{\mathrm{i}}{2}(\mathrm{Im}\, F)\chi^i_{(L)} . \tag{4.10}$$

In addition to the scalar-dependent Kähler and superpotentials, we also have to consider the gauge kinetic functions. Under the scalar isometries they transform as follows:

$$\delta(\Lambda) f_{\Lambda\Sigma} = \Lambda^\alpha k_\alpha^i \partial_i f_{\Lambda\Sigma} . \tag{4.11}$$

The part of the action that depends on the gauge kinetic function is again not invariant under such transformations. To solve this issue, one is tempted to impose the restriction $k_\alpha^i \partial_i f_{\Lambda\Sigma} = 0$. However, in the next section we will see that this lack of invariance can also be compensated by appropriate global transformations of the vector fields.

Thus far, we have shown that the isometries (4.3) leave invariant the *bosonic* part of the supergravity action, provided they satisfy the Killing equation and are supplemented by the correct Kähler transformations. On the other hand, supersymmetry constrains the global transformations of the corresponding fermions, since supersymmetry transformations and isometries commute:

$$\begin{aligned} 0 = [\delta(\Lambda), \delta(\epsilon)] z^i &= \delta(\Lambda) \left( \bar{\epsilon}_{(L)} \chi^i_{(L)} \right) - \delta(\epsilon) \left( \Lambda^\alpha k_\alpha^i \right) \\ &= \bar{\epsilon}_{(L)} \left( \delta(\Lambda) \chi^i_{(L)} \right) - \Lambda^\alpha (\partial_j k_\alpha^i) \bar{\epsilon}_{(L)} \chi^j_{(L)} , \end{aligned} \tag{4.12}$$

where we used the variation of the scalars in (3.48). Then (4.12) leads to

$$\delta(\Lambda)\chi^i_{(L)} = \Lambda^\alpha (\partial_j k_\alpha^i) \chi^j_{(L)} , \tag{4.13}$$

which is enough to show invariance of the fermionic part of the action.

The symmetries we discussed so far have a trivial representation on the vectors and their supersymmetric partners. Symmetries with a non-trivial action on the vectors also exist, though, and they will be discussed in the next section.

### Electromagnetic duality for the vectors

The kinetic action for the vector fields in a generic supergravity theory has the following universal form:

$$S_{\text{v.k.}} = \int \mathrm{d}^4x \left( \frac{1}{4} e \mathcal{I}_{\Lambda\Sigma}(\phi) F_{\mu\nu}{}^{\Lambda} F^{\mu\nu\Sigma} - \frac{1}{8} \mathcal{R}_{\Lambda\Sigma}(\phi) \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}{}^{\Lambda} F_{\rho\sigma}{}^{\Sigma} \right) . \tag{4.14}$$

The second term is often referred to as the Peccei-Quinn term, and the function $\mathcal{N}_{\Lambda\Sigma}(\phi) \equiv \mathcal{R}_{\Lambda\Sigma}(\phi) + \mathrm{i}\mathcal{I}_{\Lambda\Sigma}(\phi)$ depends nontrivially on the scalar fields, $\phi^i$, of the theory. Depending on the amount of supersymmetry, $\mathcal{N}_{\Lambda\Sigma}(\phi)$ has to satisfy certain conditions. In $4D$, $\mathcal{N} = 1$ theories, we found that the $2n$ real scalars can be arranged into $n$ complex scalars $z^i$ and their conjugates $\bar{z}^i$, and the kinetic matrix $\mathcal{N}_{\Lambda\Sigma}(\phi)$ is proportional to an antiholomorphic function $\bar{f}_{\Lambda\Sigma}(\bar{z})$. In the remainder of this section we will not have to specify the number of supercharges, but the reader can always translate to $\mathcal{N} = 1$ notations via the identification $\mathcal{N}_{\Lambda\Sigma}(\phi) = -\mathrm{i}\bar{f}_{\Lambda\Sigma}(\bar{z})$. If we substitute this identification into (4.14), we recover the $\mathcal{N} = 1$ vector kinetic terms from (3.46).

Before we go on with our discussion about the general transformations that leave $S_{\text{v.k.}}$ invariant, let us take a step back and examine the easiest theory for vectors, which is electromagnetism. In that case, the vector indices $\Lambda, \Sigma, \ldots$ take only one value, the gauge kinetic function is just a negative delta function, and the Peccei-Quinn term vanishes. As such, $S_{\text{v.k.}}$ reduces to the action in (2.11). The corresponding field equation and Bianchi identity for the electromagnetic potential were given in (2.7) with zero current, $J^{\mu}_{\text{em}} = 0$. A remarkable property of these equations is their invariance under so-called *electromagnetic duality transformations*,

$$F_{\mu\nu} \quad \leftrightarrow \quad \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \, . \tag{4.15}$$

In terms of the electric and magnetic fields, this transformation becomes

$$(\boldsymbol{E}, \boldsymbol{B}) \;\rightarrow\; (\boldsymbol{B}, -\boldsymbol{E}) \, . \tag{4.16}$$

Maxwell's vacuum equations are therefore invariant under the exchange of electric and magnetic fields. Physically, this means that in the absence of charged matter, we cannot make a distinction between electric and magnetic phenomena.[5] We stress that this duality is only an on-shell symmetry since it does not leave the Maxwell Lagrangian itself invariant. Indeed, under (4.16) the Lagrangian $\mathcal{L}_{\gamma}$ in (2.11) changes sign.

[5] This conclusion changes from the moment we include electrically charged matter. Indeed, the presence of a current $J^{\mu}_{\text{em}} \neq 0$ prevents (4.16) from being a symmetry. This can be cured though, if one adds also a "magnetic current" $k^{\mu}$ to the homogeneous equation in (2.7). The transformation (4.16) should then be supplemented by a change $(j^{\mu}, k^{\mu}) \;\rightarrow\; (k^{\mu}, -j^{\mu})$. However, this possibility is probably not realized in nature since we have never observed magnetic monopole charges.

What is the relevance of these results for our discussion about the global symmetries of supergravity? Well, it turns out that every $4D$ supergravity theory possesses a set of symmetries that is an extension of the ordinary electromagnetic duality transformations [54]. In order to see how this comes about, let us continue with a second example that is slightly more complicated than electromagnetism, but already incorporates all the basic features of a generic supergravity theory. The example we have in mind contains one complex scalar field, denoted by $z$, and one vector field. The action $S_{\text{v.k.}}$ is given by:[6]

$$S_{\text{v.k.}} = \int \mathrm{d}^4 x \left( \frac{1}{4} e(\operatorname{Im} z) F_{\mu\nu} F^{\mu\nu} - \frac{1}{8} (\operatorname{Re} z)\, \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \right) . \tag{4.17}$$

The equations of motion for the vectors can be calculated via an Euler-Lagrange variation of the Lagrangian, and they should be supplemented by the Bianchi identities:

$$\text{Bianchi id.}: \qquad \partial_{[\mu} F_{\nu\rho]} = 0\,, \tag{4.18}$$

$$\text{e.o.m.}: \qquad \partial_{[\mu} G_{\nu\rho]} = 0\,, \quad \text{with } G_{\nu\rho} \equiv \frac{1}{2} e(\operatorname{Im} z) \varepsilon_{\nu\rho\sigma\lambda} F^{\sigma\lambda} + (\operatorname{Re} z) F_{\nu\rho}\,.$$

An interesting property of this set of equations –which is the equivalent of (2.7)– is its invariance under general linear transformations $GL(2,\mathbb{R})$:

$$\begin{pmatrix} F'_{\mu\nu} \\ G'_{\mu\nu} \end{pmatrix} = \mathcal{S} \begin{pmatrix} F_{\mu\nu} \\ G_{\mu\nu} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F_{\mu\nu} \\ G_{\mu\nu} \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}\,. \tag{4.19}$$

However, if we want the new $G'_{\mu\nu}$ to be related to the new $F'_{\mu\nu}$ in the same way as $G_{\mu\nu}$ was related to $F_{\mu\nu}$ in equation (4.18), we need to restrict the allowed transformations to a subset of $GL(2,\mathbb{R})$. Moreover, a non-trivial transformation (i.e., $\mathcal{S}$ is different from the identity) is only allowed if we simultaneously transform the scalar field $z$. This can be seen as follows. Suppose we start from the new desired relation

$$G'_{\nu\rho} = \frac{1}{2} e(\operatorname{Im} z') \varepsilon_{\nu\rho\sigma\lambda} F'^{\sigma\lambda} + (\operatorname{Re} z') F'_{\nu\rho} \tag{4.20}$$

$$\Leftrightarrow \quad cF_{\mu\nu} + dG_{\mu\nu} = \frac{1}{2} e(\operatorname{Im} z') \varepsilon_{\nu\rho\sigma\lambda} \left( aF^{\rho\sigma} + bG^{\rho\sigma} \right) + (\operatorname{Re} z') \left( aF_{\mu\nu} + bG_{\mu\nu} \right)\,,$$

with $z'$ the transformed scalar field. Then the problem we need to solve can be phrased as follows: for what combinations of $a, b, c, d$ and $z'$ do we obtain a relation in the second line of (4.20) that is identical to the one in (4.18)? Let us consider

[6] This model is inspired by the action for an $\mathcal{N} = 2$ vector multiplet coupled to a complex scalar $z$ via the gauge kinetic function $\mathcal{N}_{11} = z$.

two examples. First we assume that $\mathcal{S}_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. For this choice, the second line in (4.20) becomes

$$G_{\mu\nu} = \frac{1}{2} e(\mathrm{Im}\, z')\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} + (\mathrm{Re}\, z' - 1)\, F_{\mu\nu}\,. \tag{4.21}$$

This is identical to the expression for $G_{\mu\nu}$ in (4.18) iff the scalar field transforms as $z \;\to\; z' = z + 1$. The second example we want to consider is $\mathcal{S}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. A similar analysis reveals that the scalar field has to undergo a simultaneous transformation $z \;\to\; z' = 1/z$. In fact, $\mathcal{S}_1$ and $\mathcal{S}_2$ generate a special subgroup of $GL(2,\mathbb{R})$, namely the 2-dimensional symplectic transformations. A careful analysis reveals that the set of Bianchi and field equations is invariant under all symplectic rotations, and the new 2-forms $G'_{\mu\nu}$ are always related to the $F'_{\mu\nu}$ via the relation in (4.18) and given the appropriate transformation of the scalar field. More precisely, we find that

$$\mathcal{S} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2,\mathbb{R}) \quad \text{and} \quad z' = \frac{c + dz}{a + bz}\,. \tag{4.22}$$

The symplectic condition imposes the relation $ad - bc = 1$ on the parameters in $\mathcal{S}$.

We have now determined the extension of ordinary electromagnetic duality to the easiest example of a full fledged supergravity theory with a vector kinetic action of the form (4.14). Again we stress that the transformations (4.22) are only on-shell symmetries and in general, they do not leave the action invariant:

$$\begin{aligned} S'_{\mathrm{v.k.}} &= \frac{1}{8}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma} F'_{\mu\nu} G'_{\rho\sigma} \\ &= \frac{1}{8}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma}\, (aF_{\mu\nu} + bG_{\mu\nu})\,(cF_{\rho\sigma} + dG_{\rho\sigma}) \\ &= S_{\mathrm{v.k.}} + \frac{1}{8}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma}\, (2bcF_{\mu\nu} + bdG_{\mu\nu})\, G_{\rho\sigma}\,, \end{aligned} \tag{4.23}$$

where we used the symplectic property $ad - bc = 1$ and the fact that $\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ is a total derivative. We see that (4.22) is only a true invariance of the vector kinetic action if $b = 0$. For this subset of symplectic rotations, the transformation of the scalar field simplifies to $z' = \frac{c+dz}{a}$.

Finally we remark that (4.22) is only a symmetry of the full theory if it also leaves the other terms in the action invariant. In particular, the scalar kinetic term describes a $Sp(2,\mathbb{R})/SO(2)$ scalar coset manifold which is invariant under the full isometry group $Sp(2,\mathbb{R})$ of the coset. An explicit construction yields

$$S_{\mathrm{kin,\,scalar}} = -\frac{1}{4}\int \mathrm{d}^4x\, e\frac{1}{(\mathrm{Im}\, z)^2}\partial_\mu z \partial^\mu \bar{z}\,. \tag{4.24}$$

This shows how the symmetries of the scalar sector and of the vector sector are intimately connected.

How can we now extend all these results to generic supergravities with a general vector kinetic Lagrangian $S_{\text{v.k.}}$? In order to simplify our discussion, we start with some definitions. For any real antisymmetric 2-tensor $T_{\mu\nu}$ we define (anti) self-dual tensors $T^{\pm}_{\mu\nu}$ as follows:

$$T^{\pm}_{\mu\nu} \equiv \frac{1}{2}\left(T_{\mu\nu} \pm \tilde{T}_{\mu\nu}\right), \quad \text{with} \quad \tilde{T}_{\mu\nu} \equiv -\frac{1}{2}\mathrm{i}e\varepsilon_{\mu\nu\rho\sigma}T^{\rho\sigma}. \tag{4.25}$$

With the help of this definition, one can show that the action $S_{\text{v.k.}}$ takes the following simple form:

$$S_{\text{v.k.}} = \int \mathrm{d}^4x\, \frac{1}{2} e\, \mathrm{Im}\left(F^{+\Lambda}_{\mu\nu} G^{\mu\nu\,+}_{\Lambda}\right), \tag{4.26}$$

where we have introduced the dual field strengths

$$G^{\mu\nu\,+}_{\Lambda} \equiv 2\mathrm{i}e^{-1}\frac{\partial \mathcal{L}_{v.k.}}{\partial F^{+\Lambda}_{\mu\nu}} = \mathcal{N}_{\Lambda\Sigma}F^{\mu\nu+\Sigma}. \tag{4.27}$$

The Bianchi identities and equations of motion can then be rewritten in terms of the self-dual tensors,

$$\partial_{[\mu}\, \mathrm{Im}\, F^{+\Lambda}_{\nu\rho]} = 0\,, \qquad \partial_{[\mu}\, \mathrm{Im}\, G^{+}_{\nu\rho]\Lambda} = 0\,. \tag{4.28}$$

So far we have not done anything mysterious. In the case of $\mathcal{N} = z$, all these equations reduce to expressions from the previous example. Also the next step is not new. We will study the general form of the electromagnetic duality transformations that leave $S_{\text{v.k.}}$ invariant. We begin with the observation that the combined set of Bianchi identities and equations of motion are invariant under general linear transformations $GL(2n_V,\mathbb{R})$:

$$\begin{pmatrix} F'^{+} \\ G'^{+} \end{pmatrix} = \mathcal{S}\begin{pmatrix} F^{+} \\ G^{+} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} F^{+} \\ G^{+} \end{pmatrix}. \tag{4.29}$$

This time, $A^{\Lambda}{}_{\Sigma}$, $B^{\Lambda\Sigma}$, $C_{\Lambda\Sigma}$ and $D_{\Lambda}{}^{\Sigma}$ are $n_V \times n_V$ real matrices, where $n_V$ is the number of vectors in the theory, i.e., $\Lambda, \Sigma, \ldots = 1, \ldots, n_V$. In order to simplify the notation, we will combine the field strengths $F_{\mu\nu}{}^{\Lambda}$ and their duals $G_{\mu\nu\Lambda}$ into a $2n_V$-plet, $F_{\mu\nu}{}^{M}$, such that $F^M \equiv (F^\Lambda, G_\Lambda)$. Accordingly, the matrices $\mathcal{S}$ get an upper and a lower index, $\mathcal{S} = \mathcal{S}^M{}_N$. This allows us to write (4.28) and (4.29) in the following compact way:

$$\partial_{[\mu}\, \mathrm{Im}\, F^{+}_{\nu\rho]}{}^{M} = 0\,, \qquad F'_{\mu\nu}{}^{M+} = \mathcal{S}^M{}_N F^{N+}_{\mu\nu}\,. \tag{4.30}$$

If we require that $G'_{\Lambda}{}^{\mu\nu\,+}$ can be written as the derivative of a transformed action and its relation to $F'_{\Lambda}{}^{\mu\nu\,+}$ is identical to (4.27), the gauge kinetic function should transform as

$$\mathcal{N}_{\Lambda\Sigma} \;\to\; \mathcal{N}'_{\Lambda\Sigma} = (C + D\mathcal{N})_{\Lambda\Omega}\left[(A + B\mathcal{N})^{-1}\right]^{\Omega}{}_{\Sigma}\,. \tag{4.31}$$

The new function $\mathcal{N}'_{\Lambda\Sigma}$ is only symmetric in its lower indices if the following relations for the matrices $A$, $B$, $C$ and $D$ are satisfied:

$$A^T C - C^T A = 0\,, \qquad B^T D - D^T B = 0\,, \qquad A^T D - C^T B = \mathbb{1}\,. \tag{4.32}$$

In particular we have $B^{\Lambda\Sigma} = B^{\Sigma\Lambda}$ and $C_{\Lambda\Sigma} = C_{\Sigma\Lambda}$. This is a generalization of (4.22). The relations in (4.32) express that $\mathcal{S}^M{}_N$ is a symplectic matrix and all admissible rotations $\mathcal{S}^M{}_N$ form the group $Sp(2n_V, \mathbb{R})$:

$$\mathcal{S}^T \Omega \mathcal{S} = \Omega\,, \quad \text{with} \quad \Omega_{MN} = \begin{pmatrix} 0 & \Omega_\Lambda{}^\Sigma \\ \Omega^\Lambda{}_\Sigma & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1}_\Lambda^\Sigma \\ -\mathbb{1}^\Lambda_\Sigma & 0 \end{pmatrix} \tag{4.33}$$

the symplectic metric. Furthermore, we define $\Omega^{MN}$ via $\Omega^{MN}\Omega_{NP} = -\mathbb{1}^M{}_P$. Note that the components of $\Omega^{MN}$ should not be written as $\Omega^\Lambda{}_\Sigma$ etc., as these are different from (4.33).

Thus far we have only discussed symmetries of the vector field equations and the Bianchi identities. But ultimately we are interested in the transformations that leave the total supergravity action invariant. We will focus on the purely bosonic part of the action only, whereas the symplectic structure of the full action has been explored in [60]. The variations we need to take into account are: (i) the symplectic rotations of the field strengths and their duals and (ii) the implementation of (4.31) by transformations of the scalars on which $\mathcal{N}_{\Lambda\Sigma}$ depends. First, consider a general variation of the vector kinetic Lagrangian:

$$\mathcal{L}'_{\text{v.k.}} = \frac{1}{2}\,\text{Im}\left(F'^{+\Lambda}_{\mu\nu} G'^{\mu\nu+}_\Lambda\right) = \frac{1}{2}\,\text{Im}\left(\mathcal{N}'_{\Lambda\Sigma} F'^{+\Lambda}_{\mu\nu} F'^{\mu\nu+\Sigma}\right)\,. \tag{4.34}$$

We see that electromagnetic duality transformations lead to a new Lagrangian $\mathcal{L}'_{\text{v.k.}}(F')$, which is of a similar form as the original Lagrangian $\mathcal{L}_{\text{v.k.}}(F)$, but with a new gauge kinetic function $\mathcal{N}'_{\Lambda\Sigma}$ and field strengths $F'_{\mu\nu}{}^\Lambda = 2\partial_{[\mu} A'_{\nu]}{}^\Lambda$. Although the new Lagrangian differs from the old one in its dependance on the scalar fields, its physical content is exactly the same. We say that $\mathcal{L}'_{\text{v.k.}}(F')$ gives a description of the theory "in a different duality frame".

Once we have fixed a particular duality frame, we can look for transformations that leave the action invariant. These form a subset of the symplectic transformations, such that the Lagrangian remains unchanged,

$$\mathcal{L}'_{\text{v.k.}}(F') = \mathcal{L}_{\text{v.k.}}(F)\,, \tag{4.35}$$

and (4.31) is implemented by appropriate transformations of the scalars. The condition (4.35) is only satisfied for symplectic matrices $\mathcal{S}^M{}_N$ with $B^{\Lambda\Sigma} = 0$. Then the Lagrangian is invariant up to a total derivative (if $C \neq 0$):

$$\mathcal{L}'_{\text{v.k.}}(F') \quad = \quad \mathcal{L}_{\text{v.k.}}(F) + \frac{1}{2}\,\text{Im}\left(F^{+\Lambda}_{\mu\nu} (C^T A)_{\Lambda\Sigma} F^{\mu\nu+\Sigma}\right)\,. \tag{4.36}$$

Moreover, the transformation of the gauge kinetic function simplifies, $\mathcal{N}' = (C + D\mathcal{N})A^{-1}$. Finally, if we want the symplectic rotations with $B^{\Lambda\Sigma} = 0$ to be invariances of the full supergravity action, the scalar transformations that produce $\mathcal{N}' = (C + D\mathcal{N})A^{-1}$ also have to leave the other terms in the action invariant (in particular the scalar kinetic terms). This brings us back to our discussion about scalar isometries in the previous section, from which we deduce that

$$\mathcal{N}(\phi') = [C + D\mathcal{N}(\phi)]\, A^{-1}\,, \tag{4.37}$$

where $\phi'^i = \phi^i + \Lambda^\alpha k^i_\alpha(\phi)$ to first order in the parameters, and $k^i_\alpha(\phi)$ is a Killing vector of the scalar manifold that is compatible with the Kähler structure. It is clear that in different duality frames (i.e., different matrices $\mathcal{N}(\phi)$), the action might have different sets of invariances.

### Summary & global symmetries of $\mathcal{N} = 1$ supergravity

In the previous two sections we have discussed the global internal symmetries (and invariances) of $4D$ supergravities. Symmetry transformations of the scalars are described by isometries of the scalar manifold, transformations of the vector sector are described by symplectic rotations. Both transformations have to be compatible in the sense that they should satisfy (4.31). As a result, the group of global symmetries $G_{\text{global}}$ is generally not the direct product of isometries and symplectic transformations, but only a subgroup of the latter:

$$G_{\text{global}} \subseteq Sp(2n_V, \mathbb{R}) \times \text{Iso}(\mathcal{M}_{\text{scalar}})\,. \tag{4.38}$$

This is certainly true for $\mathcal{N} = 1$, but also for $\mathcal{N} = 2$ where the role of $\text{Iso}(\mathcal{M}_{\text{scalar}})$ is played by the isometries of the hypermultiplet scalar manifold.[7] For (half)-maximal supergravity the situation is slightly different since all global symmetries can be embedded into the symplectic group. The prototypical example is $\mathcal{N} = 8$, where the scalar isometry group $E_{7(+7)}$ is embedded into the symplectic group $Sp(56, \mathbb{R})$. This is due to the fact that in (half-)maximal supergravities the vectors are supersymmetrically related to scalar fields, and therefore their global symmetries are connected to the symmetries of scalar manifolds.

Let us now focus on $\mathcal{N} = 1$ theories and collect all the results about their global symmetries. If we denote the generators of $G_{\text{global}}$ by $\delta_\alpha$, $\alpha = 1, \ldots, \dim(G_{\text{global}})$, then the representation of this group on the various (bosonic) fields can be written

[7] In $\mathcal{N} = 2$, also other scalars might be present, but these sit in the vector multiplet and are therefore part of the symplectic structure.

as:

$$\begin{aligned}
\delta(\Lambda)z^i \equiv \Lambda^\alpha\delta_\alpha z^i &= \Lambda^\alpha k^i_\alpha\,,\\
\delta(\Lambda)\begin{pmatrix}F^\Sigma\\ G_\Sigma\end{pmatrix} &= -\Lambda^\alpha\begin{pmatrix}(t_\alpha)_\Lambda{}^\Sigma & (t_\alpha)^{\Lambda\Sigma}\\ (t_\alpha)_{\Lambda\Sigma} & (t_\alpha)^\Lambda{}_\Sigma\end{pmatrix}\begin{pmatrix}F^\Lambda\\ G_\Lambda\end{pmatrix}\,,\\
\delta(\Lambda)e_\mu{}^a &= 0\,.
\end{aligned} \tag{4.39}$$

Here we used $\mathcal{S}^M{}_N \equiv \mathbb{1} - \Lambda^\alpha(t_\alpha)_N{}^M$ and $\Lambda^\alpha$ is the infinitesimal parameter corresponding to $\delta_\alpha$. The different objects in the transformation rules are subject to the following constraints:

- The holomorphic Killing vectors $k^i_\alpha$ are isometries of the scalar manifold: $\mathrm{i}g_{i\bar{\jmath}}k^{\bar{\jmath}}_\alpha = \partial_i\mathcal{P}_\alpha$.
- The scalar transformations should be compatible with the Kähler structure and lead to an appropriate transformation of the superpotential: $k^i_\alpha\partial_i W = \left(k^i_\alpha\partial_i\mathcal{K} + \mathrm{i}\mathcal{P}_\alpha\right)W$.
- The matrices $t_\alpha$ satisfy the symplectic condition: $(t_\alpha)_{[M}{}^N\Omega_{N]P} = 0$.
- The transformation of the gauge kinetic function (we use the $\mathcal{N}=1$ notation $f_{\Lambda\Sigma}$) due to its scalar dependence should be compatible with the symplectic structure: $k^i_\alpha\partial_i f_{\Lambda\Sigma} = \mathrm{i}(t_\alpha)_{\Lambda\Sigma} + 2(t_\alpha)^\Omega{}_{(\Lambda}f_{\Sigma)\Omega} + \mathrm{i}(t_\alpha)^{\Omega\Xi}f_{\Omega\Lambda}f_{\Xi\Sigma}$.

These constraints may imply that certain generators $\delta_\alpha$ have either a trivial action on the vectors (i.e., $t_\alpha = 0$) or a trivial action on the scalars (i.e., $k^i_\alpha = 0$). Another special case is when the isometry is non-trivial, but $f_{\Lambda\Sigma}$ does not transform under it, as happens, e.g., when $f_{\Lambda\Sigma} = -\mathbb{1}$ is constant. Finally, we note that the transformations in (4.39) only leave the $\mathcal{N}=1$ supergravity action invariant if $(t_\alpha)^{\Lambda\Sigma} = 0$. The invariance group of the action is therefore only a subgroup of $G_{\text{global}}$ and depends on the chosen duality frame, as we discussed at the end of §4.1.

## 4.2 Overview of conventional gaugings

We have now reviewed how the field content transforms under the global symmetry group $G_{\text{global}}$ of a basic (matter-coupled) supergravity theory. The next step will be to construct the gaugings of the theory.

Recall that we have encountered local symmetries of supergravity in two different places already. The first time was in §3.2, where we described the vector indices $\Lambda, \Sigma, \ldots$ as "gauge indices", enumerating the different generators in the gauge group $U(1)^{n_V}$. Indeed, any basic supergravity with $n_V$ vectors is invariant under the Abelian gauge transformations

$$\delta(\Lambda) A_\mu{}^\Sigma(x) = \partial_\mu \Lambda^\Sigma(x) \, , \tag{4.40}$$

with $\Lambda^\Sigma(x)$ the corresponding space-time dependent gauge parameters. All the other fields in the theory are invariant under $\delta(\Lambda)$. Our second encounter with local symmetries of ($\mathcal{N} = 1$) supergravity was in §3.3, where we discussed Kähler transformations. The corresponding gauge field $\mathcal{A}_\mu$ is an auxiliary field in the supergravity multiplet and transforms as $\delta_{\mathrm{K}} \mathcal{A}_\mu = -\partial_\mu(\operatorname{Im} F)$, with $F$ a holomorphic function of the scalar fields.

In this section we construct supergravity theories with a gauge group that is more general than the $n_V$-th power of $U(1)$. We will make use of the conventional gauging procedure, outlined in chapter 2, that starts from a symmetry of the ungauged action and introduces minimal couplings via covariant derivatives and covariant field strengths. We will see that a naive application of this procedure to supergravity theories leads to a few problems that are mainly related to (i) the presence of a Peccei-Quinn term in (4.14) and (ii) supersymmetry invariance. An adequate solution requires additional modifications to both the action and the transformations rules, to be discussed in §4.2.

### Basic ingredients

Suppose we want to gauge a subgroup $G_0 \subset G_{\mathrm{global}}$. For the conventional gaugings we will discuss here, this is only possible if the global transformations in $G_0$ leave the Lagrangian invariant. In other words, $G_0$ has to act on $F_{\mu\nu}{}^M = (F_{\mu\nu}{}^\Lambda, G_{\mu\nu\,\Lambda})$ by lower block triangular matrices (i.e. those with $B^{\Lambda\Sigma} = 0$).[8] Therefore, the first step in the gauging of $G_0$ will always be to switch to a symplectic duality frame in which $G_0$ acts on $F_{\mu\nu}{}^M$ in the right way. Once the appropriate duality frame has been selected, we can proceed as usual. We need to introduce gauge covariant derivatives and covariant field strengths, where the role of the gauge fields is played by the vectors $A_\mu{}^\Lambda$, $\Lambda = 1, \ldots, n_V$ in the theory.

A famous example is the $\mathcal{N} = 8$ Lagrangian with global invariance group $S\ell(8,\mathbb{R})$ and 28 vectors that transform in the antisymmetric **28** representation of $S\ell(8,\mathbb{R})$, as described in [46]. Part of $S\ell(8,\mathbb{R})$ can be gauged using the 28 vectors as gauge fields. The first example of such a gauging was given in [47], where a $G_0 = \mathrm{SO}(8)$ subgroup of $S\ell(8,\mathbb{R})$ was promoted to a local symmetry group. Later on, more general examples were constructed. Starting from the Lagrangian that has $S\ell(8,\mathbb{R})$ as its invariance group, the most general gaugings that can be constructed are based on the so-called $\mathrm{CSO(p,q,r)}$ groups, where $p + q + r = 8$ [62–64]. Other

[8] In the context of symplectically covariant gaugings [61], however, this restriction can be lifted, and we will come back to these in §4.3.

gaugings of $\mathcal{N}=8$ supergravity have also been found, starting from a Lagrangian in a different duality frame and therefore with a different invariance group, see e.g. [65].

In general, we have to associate a vector to each generator in $G_0$. Therefore, the dimension of $G_0$ cannot exceed the number of available vectors.[9] It is convenient to include also the $U(1)$ transformations from (4.40), such that $\dim(G_0) = n_V$ always. We will denote the generators of $G_0$ with $\delta_\Lambda$ and since $G_0$ is a subgroup of $G_{\text{global}}$, each of them can be written as a linear combination of $\delta_\alpha$-generators:

$$\delta_\Lambda \equiv \Theta_\Lambda{}^\alpha \delta_\alpha \,. \tag{4.41}$$

Here we introduced the so-called "embedding tensor"[10], denoted by $\Theta_\Lambda{}^\alpha$. One can think of it as a $n_V \times \dim(G_{\text{global}})$ dimensional matrix with constant entries. Of course the generators $\delta_\Lambda$ should form a group structure with structure constants $f_{\Lambda\Sigma}{}^\Omega$ and therefore we find the following quadratic constraint on the entries of the embedding tensor:

$$\Theta_{[\Lambda}{}^\alpha \Theta_{\Sigma]}{}^\beta f_{\alpha\beta}{}^\gamma = f_{\Lambda\Sigma}{}^\Omega \Theta_\Omega{}^\gamma \,. \tag{4.42}$$

From (4.39) and (4.41) we see that under the elements of $G_0$, the symplectic vector $F_{\mu\nu}{}^M$ transforms as

$$\delta_\Lambda F_{\mu\nu}{}^M = \Theta_\Lambda{}^\alpha \delta_\alpha F_{\mu\nu}{}^M = -\Theta_\Lambda{}^\alpha (t_\alpha)_N{}^M F_{\mu\nu}{}^N \equiv -X_{\Lambda N}{}^M F_{\mu\nu}{}^N \,, \tag{4.43}$$

where we introduced the new notation $X_{\Lambda N}{}^M \equiv \Theta_\Lambda{}^\alpha (t_\alpha)_N{}^M$. The matrix $(X_\Lambda)_N{}^M$ satisfies the symplectic condition (4.33) because $(t_\alpha)_N{}^M$ does. If we write equation (4.43) in its different components, we find

$$\delta_\Lambda \begin{pmatrix} F_{\mu\nu}{}^\Sigma \\ G_{\mu\nu\Sigma} \end{pmatrix} = -\begin{pmatrix} X_{\Lambda\Omega}{}^\Sigma & 0 \\ X_{\Lambda\Omega\Sigma} & X_\Lambda{}^\Omega{}_\Sigma \end{pmatrix} \begin{pmatrix} F_{\mu\nu}{}^\Omega \\ G_{\mu\nu\Omega} \end{pmatrix} \,. \tag{4.44}$$

Because the field strengths need to transform in the adjoint representation of the gauge group, we demand that $X_{\Lambda\Omega}{}^\Sigma = f_{\Lambda\Omega}{}^\Sigma$, which means that the symmetric part of $X_{\Lambda\Omega}{}^\Sigma$ should vanish:

$$X_{(\Lambda\Omega)}{}^\Sigma = \Theta_{(\Lambda}{}^\alpha (t_\alpha)_{\Omega)}{}^\Sigma = 0 \,. \tag{4.45}$$

This is yet another constraint on the embedding tensor, although this time it is linear in $\Theta_\Lambda{}^\alpha$, as opposed to (4.42). The symplectic properties of $(X_\Lambda)_N{}^M$ also guarantee that $X_\Lambda{}^\Omega{}_\Sigma = -f_{\Lambda\Sigma}{}^\Omega$ and $X_{\Lambda\Omega\Sigma} = X_{\Lambda(\Omega\Sigma)}$.[11] Under these restrictions, the transformation of the kinetic matrix $\mathcal{N}_{\Lambda\Sigma}$ becomes

$$\delta(\Lambda)\mathcal{N}_{\Lambda\Sigma} = \Lambda^\Xi \left( -X_{\Xi\Lambda\Sigma} + 2 f_{\Xi(\Sigma}{}^\Omega \mathcal{N}_{\Lambda)\Omega} \right) \,. \tag{4.48}$$

[9] In the examples above, the dimension of both $SO(8)$ and $\mathrm{CSO(p,q,r)}$ is 28, which is equal to the number of available gauge fields.

[10] The embedding tensor will play an important role in the classification of all possible gaugings in §4.3. We have already introduced it here for later convenience.

[11] In theories with extended supersymmetry, the index structure of $X_{\Lambda\Omega\Sigma}$ is further constrained in symplectic bases with a prepotential. This can most easily be demonstrated

Moreover, the condition (4.42) on the embedding tensor reduces to

$$\Theta_{[\Lambda}{}^{\alpha}\Theta_{\Sigma]}{}^{\beta}f_{\alpha\beta}{}^{\gamma} - \Theta_{[\Lambda}{}^{\alpha}(t_\alpha)_{\Sigma]}{}^{\Omega}\Theta_{\Omega}{}^{\gamma} = 0\,. \tag{4.49}$$

Furthermore, the transformation (4.48) and the closure relation (4.49) get intertwined if we require that $[\delta_\Omega, \delta_\Theta]\mathcal{N}_{\Lambda\Sigma} = f_{\Omega\Theta}{}^{\Xi}\delta_\Xi\mathcal{N}_{\Lambda\Sigma}$. This requirement is equivalent to the usual Jacobi identity and an extra consistency condition:

$$f_{\Xi[\Lambda}{}^{\Theta}f_{\Sigma\Omega]}{}^{\Xi} = 0\,, \tag{4.50}$$

$$f_{\Omega\Theta}{}^{\Xi}X_{\Xi\Lambda\Sigma} + 2f_{\Lambda[\Theta}{}^{\Xi}X_{\Omega]\Sigma\Xi} + 2f_{\Sigma[\Omega}{}^{\Xi}X_{\Omega]\Lambda\Xi} = 0\,. \tag{4.51}$$

Given all these ingredients, we are finally ready to proceed to the actual gauging of $G_0$. For each generator $\delta_\Sigma$ we introduce a space-time dependent parameter $\Lambda^\Sigma(x)$ that appears with a derivative in the gauge transformation of the vector fields:

$$\delta(\Lambda)A_\mu{}^{\Sigma} = \partial_\mu\Lambda^\Sigma(x) - \Lambda^\Omega(x)A_\mu{}^{\Lambda}f_{\Omega\Lambda}{}^{\Sigma}\,. \tag{4.52}$$

This is identical to (2.48) in chapter 2, up to a rescaling of the parameters $\Lambda^\Sigma(x)$ with a gauge coupling constant $g$. We have then absorbed $g$ into the structure constants $f_{\Omega\Lambda}{}^{\Sigma}$ that depend linearly on the embedding tensor, $f_{\Omega\Lambda}{}^{\Sigma} = \Theta_\Lambda{}^{\alpha}(t_\alpha)_\Lambda{}^{\Sigma}$. Thus, the embedding tensor contains the gauge coupling constant in a linear way.

The gauge covariant derivatives take the form

$$D_\mu \equiv \partial_\mu - A_\mu{}^{\Lambda}\delta_\Lambda = \partial_\mu - A_\mu{}^{\Lambda}\Theta_\Lambda{}^{\alpha}\delta_\alpha\,, \tag{4.53}$$

where $\delta_\Lambda$ works in a suitable representation of the matter fields. For example,

$$D_\mu z^i = \partial_\mu z^i - A_\mu{}^{\Lambda}k^i_\Lambda\,, \tag{4.54}$$

$$D_\mu\lambda^\Sigma = \partial_\mu\lambda^\Sigma - f_{\Lambda\Omega}{}^{\Sigma}A_\mu{}^{\Lambda}\lambda^\Omega\,, \tag{4.55}$$

where we used the notation $k^i_\Lambda \equiv \Theta_\Lambda{}^{\alpha}k^i_\alpha$ and the gaugino's transform in the adjoint representation of the gauge group. Finally, the covariant field strengths are (recall (2.55))

$$\mathcal{F}_{\mu\nu}{}^{\Omega} \equiv \partial_\mu A_\nu{}^{\Omega} - \partial_\nu A_\mu{}^{\Omega} + f_{\Lambda\Sigma}{}^{\Omega}A_\mu{}^{\Lambda}A_\mu{}^{\Sigma}\,. \tag{4.56}$$

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for $\mathcal{N}=2$ theories, where the gauge kinetic matrix depends on the complex scalars $X^\Lambda$ of the vector multiplets. These transform in the adjoint representation, which implies

$$\delta(\Lambda)\mathcal{N}_{\Lambda\Sigma}(X) = -\Lambda^\Gamma X^\Xi\mathcal{N}_{\Gamma\Xi}{}^{\Omega}\partial_\Omega\mathcal{N}_{\Lambda\Sigma}(X)\,. \tag{4.46}$$

Hence, this gives, from (4.48),

$$X_{\Lambda\Sigma\Omega} = X^\Gamma f_{\Lambda\Gamma}{}^{\Xi}\partial_\Xi\mathcal{N}_{\Sigma\Omega} + 2f_{\Lambda(\Sigma}{}^{\Gamma}\mathcal{N}_{\Omega)\Gamma}\,, \tag{4.47}$$

which leads to $X_{\Lambda\Sigma\Omega}X^\Lambda X^\Sigma X^\Omega = 0$. As the scalars $X^\Lambda$ are independent in the presence of a prepotential [60, 66], this implies that $X_{(\Lambda\Sigma\Omega)} = 0$. An analogous argument for symplectic bases without a prepotential is missing.

### Gauge invariant action

At the beginning of the previous section we have selected a duality frame such that the (ungauged) action is invariant under global transformations $G_0$. According to our discussion in §2.2, this action becomes manifestly invariant under *local transformations* $\delta(\Lambda) = \Lambda^{\Sigma}(x)\delta_{\Sigma}$ if one makes the replacements $\partial_\mu \to D_\mu$ and ${F_{\mu\nu}}^{\Lambda} \to {\mathcal{F}_{\mu\nu}}^{\Lambda}$. Here we find the following expression for the bosonic part of the kinetic terms:

$$\begin{aligned} S_{\text{g.k.}} \;&=\; \int \mathrm{d}^4x\, e \left( \frac{1}{2}R(e) - G_{ij}\hat{D}_\mu\phi^i\hat{D}^\mu\phi^j \right. \\ &\qquad \left. + \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}{\mathcal{F}_{\mu\nu}}^{\Lambda}\mathcal{F}^{\mu\nu\Sigma} - \frac{1}{8}\mathcal{R}_{\Lambda\Sigma}e^{-1}\varepsilon^{\mu\nu\rho\sigma}{\mathcal{F}_{\mu\nu}}^{\Lambda}{\mathcal{F}_{\rho\sigma}}^{\Sigma} \right), \end{aligned} \tag{4.57}$$

where the derivatives $\hat{D}_\mu$ are now covariant with respect to all local symmetries, including local Lorentz transformations and isometries. Likewise, the fermions in the theory are minimally coupled to the gauge fields via covariant derivatives.

However, the minimal couplings one introduces in this way give rise to two complications:[12]

(i) The new action is not supersymmetric under the transformations in (3.48). In other words, the minimal couplings break supersymmetry explicitly.

(ii) The Peccei-Quinn term in $S_{\text{g.k.}}$ has a non-trivial gauge transformation under (4.48) and (4.52),

$$\begin{aligned} \delta(\Lambda)\int \mathrm{d}^4x \left( -\frac{1}{8}\mathcal{R}_{\Lambda\Sigma}\varepsilon^{\mu\nu\rho\sigma}{\mathcal{F}_{\mu\nu}}^{\Lambda}{\mathcal{F}_{\rho\sigma}}^{\Sigma} \right) \\ = \frac{1}{8}X_{\Omega\Lambda\Sigma}\int \mathrm{d}^4x \left( \Lambda^{\Omega}(x)\varepsilon^{\mu\nu\rho\sigma}{\mathcal{F}_{\mu\nu}}^{\Lambda}{\mathcal{F}_{\rho\sigma}}^{\Sigma} \right). \end{aligned} \tag{4.58}$$

For constant parameters $\Lambda^{\Omega}$ this variation vanishes, but for local parameters $\Lambda^{\Omega}(x)$ it does not. Hence, the action is not gauge invariant either.

The first issue above can be solved by adding extra parts to the original supergravity action and the supersymmetry transformations (3.48) of the fermions. In particular, all ordinary derivatives in the transformations need to be replaced by covariant derivatives. Due to this “covariantization”, also the algebra of commutators is modified. Indeed, in the presence of a gauged internal symmetry,

[12] These issues have their origin in the interplay between supersymmetry and gauged transformations. They are not specific to supergravity theories, but also appear in theories with global supersymmetry.

the right hand side of the commutator (3.44) requires an extra term that is proportional to a local variation $\delta(\Lambda)$:

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \delta_{\rm gct}(\xi) - \delta_{\rm ll}(\lambda_3) - \delta_{\rm susy}(\epsilon_3) - \delta(\Lambda_3) + \dots \,, \tag{4.59}$$

with $\Lambda_3^\Sigma = A_\mu{}^\Sigma \xi^\mu$. The dots in (4.59) denote other possible contributions to the algebra, such as a central charge. At the level of the action, one has to introduce a potential for the scalar fields that depends quadratically on the embedding tensor (and hence the gauge coupling constant). A detailed description of these changes for minimal supergravities will be postponed until chapter 5, where they are presented in combination with our research results from [1]. For $\mathcal{N} = 2$ theories, the final action and transformation rules can be found in [41]; $\mathcal{N} = 4$ supergravities have been discussed in [67,68]; and a concise description of gauged $\mathcal{N} = 8$ theories was given in [65].

The second issue above has its origin in the appearance of a constant tensor $X_{\Lambda\Sigma\Omega}$ in the transformation of the gauge kinetic function (4.48). If we set $X_{\Lambda\Sigma\Omega} = 0$, the gauge variation of the Peccei-Quinn term in (4.58) vanishes and the action is gauge invariant again. However, there is also a more general solution possible, that was first discussed for $\mathcal{N} = 2$ supergravities in [41] and later for $\mathcal{N} = 1$ supersymmetric theories in [12]. In each case, the tensor $X_{\Lambda\Sigma\Omega}$ is non-trivial, but it has to satisfy the constraint[13] $X_{(\Lambda\Sigma\Omega)} = 0$. This is immediate for $\mathcal{N} = 2$ supergravity (recall footnote 11), but should be imposed as a separate condition in $\mathcal{N} = 1$ theories. Then it was shown in [12,41] that gauge invariance of the action can be restored if one adds generalized Chern-Simons terms of the form $A \wedge A \wedge dA$ and $A \wedge A \wedge A \wedge A$. Since these results were presented for $\mathcal{N} = 1$ supergravity theories in our paper [1], we will again postpone a detailed discussion until chapter 5.

We want to finish this section with a few comments about the equations of motion for the gauge fields $A_\mu{}^\Sigma$. The latter appear non-linearly in the kinetic terms and the Chern-Simons term, whereas in all other terms of the supergravity action, they couple linearly to the matter fields. After a short calculation one finds the following general form for the field equation:

$$\varepsilon^{\mu\nu\rho\sigma} D_\nu \mathcal{G}_{\rho\sigma\Sigma} = J^\mu{}_\Sigma \,, \tag{4.60}$$

where $\mathcal{G}_{\rho\sigma\Lambda} \equiv \frac{1}{2} e \mathcal{I}_{\Lambda\Sigma} \varepsilon_{\rho\sigma\mu\nu} \mathcal{F}^{\mu\nu\Sigma} + \mathcal{R}_{\Lambda\Sigma} \mathcal{F}_{\mu\nu}{}^\Sigma$ is the non-Abelian generalization of $G_{\rho\sigma\Lambda}$, and $J^\mu{}_\Sigma \equiv \frac{\delta \mathcal{L}_{\rm matter}}{\delta A_\mu{}^\Sigma}$ is the matter current. Remark that the field equation in (4.60) has a form that is very similar to the one we found for Yang-Mills gauge theories in (2.64). Together with the Bianchi identity $\varepsilon^{\mu\nu\rho\sigma} D_\nu \mathcal{F}_{\rho\sigma}{}^\Sigma = 0$, equation (4.60) fixes the dynamics of the vector fields. However, this time the set of field equations and Bianchi identities is not invariant under symplectic rotations, as

[13] Also this constraint can be lifted if we include quantum effects. Then the classical non-invariance of the action that is proportional to $X_{(\Lambda\Sigma\Omega)}$ can be used to cancel the gauge quantum anomaly. This mechanism will be discussed at length in chapter 5.

opposed to the ungauged theory, and we conclude that the conventional gauging method breaks electromagnetic duality covariance explicitly.

## 4.3 Introduction to generalized gaugings

Conventional gauging methods require the invariance of the action under the global symmetries we want to gauge. However, in 4 dimensions, not every global symmetry of the theory automatically leaves the action invariant, and therefore we first need to make a rotation to the appropriate electromagnetic duality frame. This has been discussed extensively in the previous section. Moreover, we saw that after the gauging, duality covariance is generally lost due to the minimal couplings to electric vector fields only. These imperfections make it difficult to pursue a systematic analysis of all possible gaugings using the conventional methods.

The first attempt to systematically classify all possible gaugings of a given ungauged supergravity was presented in [69] for $\mathcal{N}=8$. Later on, this treatment was reexamined and formalized [70], which led to the construction of a new and powerful formalism, called the embedding tensor mechanism. It was first used in $3D$ by H. Samtleben and H. Nicolai [71, 72], and later extended to higher dimensions in [61, 70, 73, 74]. Its domain of application is not restricted to supergravity theories, despite the fact that its main use is in this context. The progress is rather related to a better understanding of all the possibilities of gauge groups and their coupling to all the fields in the theories. The virtues of this formalism in 4 dimensions are therefore (i) its capacity to classify all possible gaugings of generic field theories, without the need to investigate different duality frames; (ii) a powerful method to construct Lagrangians that are invariant under the most general gauge groups and (iii) its manifest electromagnetic covariant form.

The key ingredient of the formalism is an extension of the original embedding tensor $\Theta_\Lambda{}^\alpha$ (introduced in equation (4.41)) to the more general object $\Theta_M{}^\alpha = (\Theta_\Lambda{}^\alpha, \Theta^{\Lambda\alpha})$, where $M$ is a symplectic index. In contrast to $\Theta_\Lambda{}^\alpha$ that was used to couple global symmetry generators $\delta_\alpha$ to electric vectors $A_\mu{}^\Lambda$ in the covariant derivatives (4.53), the extended embedding tensor will couple the generators $\delta_\alpha$ to both electric ánd magnetic vectors[14], collectively denoted by $A_\mu{}^M = (A_\mu{}^\Lambda, A_{\mu\Lambda})$:

$$D_\mu \equiv \partial_\mu - A_\mu{}^M \Theta_M{}^\alpha \delta_\alpha \,. \tag{4.62}$$

This definition also clarifies the role of the embedding tensor as a matrix of deformation parameters (or coupling constants). For example, in the case of

[14] Remember that magnetic vectors $A_{\mu\Lambda}$ are defined via the equations of motion, i.e.,

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\nu G_{\rho\sigma\Lambda} = 0 \quad \rightarrow \quad G_{\rho\sigma\Lambda} \equiv 2\partial_{[\rho} A_{\sigma]\Lambda} \,. \tag{4.61}$$

a simple gauge group it is proportional to the conventional gauge coupling constant $g$.

In the remainder of this section, our first goal will be to obtain a better understanding of the concept "magnetic gauging", i.e., we want to clarify how the magnetic vector fields can be used to gauge global symmetries. From this discussion it will become clear that the definition in (4.62) needs to be supplemented by an additional structure that turns the embedding tensor formalism into a consistent framework. These additional ingredients will be reviewed in §4.3. The reader should accept that we restrict our discussion to a formal development of the formalism, without studying the interesting results that can be obtained from it. For this, we refer to the literature [70, 73–85].

### Magnetic gaugings

Consider the following Lagrangian, describing a real scalar field $\phi$ and a vector $A_\mu$:

$$\mathcal{L}_0 = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \,. \tag{4.63}$$

This Lagrangian is invariant under the following independent global transformations:

$$\delta(\theta)\phi = \theta \,, \qquad \delta(\Xi)A_\mu = -\Xi_\mu \,. \tag{4.64}$$

Here, $\theta$ and $\Xi_\mu$ are both constant parameters. The shift symmetry of the scalar field can be gauged using the vector $A_\mu$ as a gauge field. This boils down to replacing ordinary derivatives $\partial_\mu\phi$ by covariant ones,

$$D_\mu\phi \equiv \partial_\mu\phi - A_\mu \,. \tag{4.65}$$

Then the new Lagrangian is invariant under the gauge transformations $\delta\phi = \theta(x)$ and $\delta A_\mu = \partial_\mu\theta(x)$. This procedure is familiar to the reader.

What happens if we want to use the *magnetic* vector field[15] $\tilde{A}_\mu$ as the gauge field? Consider the replacement $\partial_\mu\phi \to \tilde{D}_\mu\phi = \partial_\mu\phi - \tilde{A}_\mu$ in the action. Then the variation of $\mathcal{L} \equiv \mathcal{L}_0|_{\partial_\mu \to \tilde{D}_\mu}$ due to an arbitrary transformation of $\tilde{A}_\mu$ has the form

$$\delta\mathcal{L} = \left(-\tilde{D}_\mu\phi\right)\delta\tilde{A}^\mu \,. \tag{4.67}$$

This looks like the beginning of a duality relation between the scalar field $\phi$ and some 2-form $B_{\mu\nu}$. Such a duality can be made manifest via the addition of a

[15] We will denote the magnetic vector with $\tilde{A}_\mu$; it is defined via

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\nu G_{\rho\sigma} = 0 \quad \to \quad G_{\rho\sigma} = \varepsilon_{\rho\sigma\mu\nu}F^{\mu\nu} \equiv 2\partial_{[\rho}\tilde{A}_{\sigma]} \,. \tag{4.66}$$

topological term to the action:

$$\mathcal{L} \quad \rightarrow \quad \mathcal{L}' = \mathcal{L} + \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\tilde{A}_\mu \partial_\nu B_{\rho\sigma}\,,$$

$$\text{and} \quad \delta\mathcal{L}' = \left(-\tilde{D}^\mu\phi + \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma}\right)\delta\tilde{A}_\mu\,. \tag{4.68}$$

Now the expression between the brackets in the second line is exactly the duality relation between a scalar and a 2-form in 4 space-time dimensions.[16] We also see that the field equation of the 2-form requires that $\tilde{A}_\mu$ is pure gauge, $\varepsilon^{\mu\nu\rho\sigma}\partial_\rho\tilde{A}_\sigma = 0$, which means it is physically equivalent to a vanishing field. In order to transfer the degrees of freedom from $A_\mu$ to $\tilde{A}_\mu$, we have to gauge the $\Xi_\mu$-symmetry in (4.64). We define a new field strength $H_{\mu\nu} = F_{\mu\nu} + B_{\mu\nu}$ and replace $F_{\mu\nu} \rightarrow H_{\mu\nu}$ in the action. We find

$$\mathcal{L}'' = -\frac{1}{2}\tilde{D}_\mu\phi\tilde{D}^\mu\phi - \frac{1}{4}H_{\mu\nu}H^{\mu\nu} + \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\tilde{A}_\mu\partial_\nu B_{\rho\sigma}\,. \tag{4.69}$$

Then $\mathcal{L}''$ is invariant (up to a total derivative) under the following set of local field transformations,

$$\delta A_\mu = \partial_\mu\theta(x) - \Xi_\mu(x)\,, \qquad \delta\tilde{A}_\mu = \partial_\mu\tilde{\theta}(x)\,,$$

$$\delta\phi = \tilde{\theta}(x)\,, \qquad \delta B_{\mu\nu} = 2\partial_{[\mu}\Xi_{\nu]}(x)\,. \tag{4.70}$$

The Lagrangian $\mathcal{L}''$ is said to describe a *magnetic gauging* of the global symmetries in (4.64).

Of course, this is a rather artificial construction (or a trivial example of this construction), since one can always integrate out the $B_{\mu\nu}$-field using its equation of motion, $B_{\mu\nu} = -F_{\mu\nu} + \varepsilon_{\mu\nu\rho\sigma}\partial^\rho\tilde{A}^\sigma$. This in return gives a Lagrangian that is equivalent to the one obtained via the electric gauging:

$$\mathcal{L}'' = -\tilde{D}_\mu\phi\tilde{D}^\mu\phi - \frac{1}{4}G_{\mu\nu}G^{\mu\nu}\,, \qquad G_{\mu\nu} \equiv 2\partial_{[\mu}\tilde{A}_{\nu]}\,. \tag{4.71}$$

Let us now consider a slightly more complicated example. We introduce one complex scalar field $z$, and one vector field. The scalars $z$, $\bar{z}$ are the coordinates on a $SL(2,\mathbb{R})/SO(2)$ coset manifold, and the gauge kinetic function only depends on the imaginary part of $z$, $\mathcal{N}_{11} = -\mathrm{i}\,\mathrm{Im}(z)$. Then the ungauged Lagrangian takes the following form:[17]

$$\mathcal{L} = -\frac{1}{4(\mathrm{Im}\,z)^2}\partial_\mu z\partial^\mu\bar{z} - \frac{1}{4}(\mathrm{Im}\,z)F_{\mu\nu}F^{\mu\nu}\,. \tag{4.72}$$

[16] If we integrate out the magnetic vector field, the new Lagrangian describes the same physics as $\mathcal{L}_0$, but the scalar degrees of freedom are reorganized into the 2-forms.

[17] This model does not describe the bosonic part of a supergravity action since its form is inconsistent with supersymmetry. Rather, we will use it as an easy example to illustrate some non-trivial properties of the magnetic gauging procedure.

Because $\mathcal{N}_{11}$ has no real part, the Peccei-Quinn term is absent. Apart from the electromagnetic duality transformations[18] that cannot be gauged because the vectors themselves are charged, the Lagrangian (4.72) possesses a global symmetry that works with a shift on the real part of $z$:

$$\delta \operatorname{Re} z = \theta \,. \tag{4.74}$$

This is the equivalent of (4.64), and since the vectors are not charged under it, it is possible to gauge this symmetry. We will immediately jump ahead to the magnetic gauging, and follow the steps that were outlined in the previous example. First we introduce a covariant derivative $\tilde{D}_\mu z = \partial_\mu z - \tilde{A}_\mu$. Since $\tilde{A}_\mu$ is a real field, this boils down to $\tilde{D}_\mu(\operatorname{Im} z) = \partial_\mu(\operatorname{Im} z)$ and $\tilde{D}_\mu(\operatorname{Re} z) = \partial_\mu(\operatorname{Re} z) - \tilde{A}_\mu$. Next, we add to the Lagrangian a topological term that couples $\tilde{A}_\mu$ to a 2-form $B_{\mu\nu}$. Finally, we introduce a modified field strength $H_{\mu\nu} = F_{\mu\nu} + B_{\mu\nu}$. Then the new Lagrangian takes the form

$$\mathcal{L}'' = -\frac{1}{4(\operatorname{Im} z)^2}\tilde{D}_\mu z \tilde{D}^\mu \bar{z} - \frac{1}{4}(\operatorname{Im} z) H_{\mu\nu}H^{\mu\nu} + \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\tilde{A}_\mu \partial_\nu B_{\rho\sigma} \,, \tag{4.75}$$

which is invariant under the local symmetries

$$\begin{aligned} &\delta A_\mu = \partial_\mu \theta(x) - \Sigma_\mu(x) \,, \qquad \delta \tilde{A}_\mu = \partial_\mu \tilde{\theta}(x) \,, \\ &\delta(\operatorname{Re} z) = \tilde{\theta}(x) \,, \qquad \delta B_{\mu\nu} = 2\partial_{[\mu}\Xi_{\nu]}(x) \,. \end{aligned} \tag{4.76}$$

However this time, integrating out the 2-form leads to an expression that is not equivalent to the one that is obtained from an electric gauging,

$$\mathcal{L}'' = -\frac{1}{4(\operatorname{Im} z)^2}\tilde{D}_\mu z \tilde{D}^\mu \bar{z} - \frac{1}{4(\operatorname{Im} z)} G_{\mu\nu}G^{\mu\nu} \,, \qquad G_{\mu\nu} \equiv 2\partial_{[\mu}\tilde{A}_{\nu]} \,. \tag{4.77}$$

Instead, we find that the Lagrangians with electric and magnetic gauging are related via $(\operatorname{Im} z)F_{\mu\nu}F^{\mu\nu} \leftrightarrow \frac{1}{\operatorname{Im} z}F_{\mu\nu}F^{\mu\nu}$. In other words, $\mathcal{L}''$ does not describe a continuous deformations of the original ungauged theory. This should not be seen as an inconsistency of the construction, but it emphasizes that electric and magnetic gaugings lead to genuinely different theories. Once we will use magnetic gaugings in the context of the embedding tensor formalism, we will see that they are equivalent to *electric gaugings in a different duality frame* [61].

[18] Because the real part of $\mathcal{N}_{11}$ is absent, the electromagnetic duality transformations do not generate the full symplectic group, but are restricted to the following form:

$$\delta \begin{pmatrix} F_{\mu\nu} \\ G_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix} \begin{pmatrix} F_{\mu\nu} \\ G_{\mu\nu} \end{pmatrix} \,, \qquad \delta z = -2\Lambda z \,, \tag{4.73}$$

with $G_{\mu\nu} \equiv \frac{1}{2}(\operatorname{Im} z)\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ the dual field strength and $\Lambda$ a constant parameter. If we had chosen a holomorphic function $\mathcal{N}_{11} = -z$, there is a Peccei-Quinn term in the Lagrangian, and the duality transformations are enhanced to those of the $Sp(2,\mathbb{R})$ group. In this way, we recover our supergravity example from §4.1.

**General structure of the embedding tensor formalism**

A priori, electric versus magnetic gaugings and the gauging of electromagnetic duality symmetries are unrelated. However, in this section we will entangle them in the electromagnetic covariant embedding tensor formalism. Our discussion is merely a recapitulation of the results that were originally presented in [61]. We refer to the literature for more details, e.g. one can consult the excellent review [86].

Recall that the basic supergravities in $4D$ have a global symmetry group that is contained in the product of the symplectic duality group and the isometry group of the scalar manifold:

$$G_{\rm global} \subseteq {\rm Sp}(2n_V, \mathbb{R}) \times {\rm Iso}(\mathcal{M}_{\rm scalar})\,. \tag{4.78}$$

In order to gauge an arbitrary subgroup, $G_0 \subset G_{\rm global}$, the $2n_V$-dimensional vector space spanned by the electric and magnetic vector fields ${A_\mu}^M = ({A_\mu}^\Lambda, A_{\mu\Lambda})$ has to be projected onto the Lie algebra of $G_0$, which is formally done with the (extended) embedding tensor ${\Theta_M}^\alpha = ({\Theta_\Lambda}^\alpha, \Theta^{\Lambda\,\alpha})$. Equivalently, ${\Theta_M}^\alpha$ completely determines the gauge group $G_0$ via the decomposition of the gauge group generators, which we will denote by $\delta_M$, into the generators of the global invariance group $G_{\rm global}$:

$$\delta_M \equiv {\Theta_M}^\alpha \delta_\alpha. \tag{4.79}$$

For each local generator $\delta_M$ we introduce a corresponding space-time dependent parameter $\Lambda^M(x)$, such that on the matter fields, the gauge group $G_0$ acts with infinitesimal transformations $\delta(\Lambda) \equiv \Lambda^M \delta_M$.

As we announced in equation (4.62), the gauge generators $\delta_M$ enter the gauge covariant derivatives of matter fields,

$$D_\mu \equiv \partial_\mu - {A_\mu}^M \delta_M = \partial_\mu - {A_\mu}^\Lambda {\Theta_\Lambda}^\alpha \delta_\alpha - A_{\mu\Lambda}\Theta^{\Lambda\alpha}\delta_\alpha\,, \tag{4.80}$$

where the generators $\delta_\alpha$ are meant to either act as representation matrices on the fermions or as Killing vectors on the scalar fields (recall §4.1). On the field strengths of the vector potentials, the generators $\delta_\alpha$ act by multiplication with the matrices $(t_\alpha)_N{}^P$, so that (4.79) is represented by matrices $(X_M)_N{}^P$ whose elements we denote as $X_{MN}{}^P$:

$$\delta_N {F_{\mu\nu}}^M = {\Theta_N}^\alpha \delta_\alpha {F_{\mu\nu}}^M = -{\Theta_N}^\alpha (t_\alpha)_P{}^M {F_{\mu\nu}}^P \equiv -X_{NP}{}^M {F_{\mu\nu}}^P\,. \tag{4.81}$$

This is a generalization of (4.43). The symplectic property (4.33) implies

$$X_{M[N}{}^Q \Omega_{P]Q} = 0\,, \qquad X_{MQ}{}^{[N}\Omega^{P]Q} = 0\,. \tag{4.82}$$

We should also stress that in this section, the symplectic matrix $(t_\alpha)_P{}^M$ will always be the most general one, and its upper-right entry $(t_\alpha)^{\Lambda\Sigma}$ can be non-zero, as opposed to (4.44). This also means we have the more general transformation of the gauge kinetic function (compare to (4.48)):

$$\delta(\Lambda)\mathcal{N}_{\Lambda\Sigma} = \Lambda^M \left(-X_{M\Lambda\Sigma} + 2X_{M(\Lambda}{}^{\Gamma}\mathcal{N}_{\Sigma)\Gamma} + \mathcal{N}_{\Lambda\Gamma}X_M{}^{\Gamma\Xi}\mathcal{N}_{\Xi\Sigma}\right) . \tag{4.83}$$

Finally, the gauge generators $\delta_M$ act on the gauge fields $A_\mu{}^M$ with a derivative on the gauge parameters,

$$\delta(\Lambda)A_\mu{}^M = \partial_\mu\Lambda^M(x) + A_\mu{}^N\Lambda^P(x)X_{NP}{}^M \equiv D_\mu\Lambda^M(x) . \tag{4.84}$$

### Constraints and extended gauge transformations

The embedding tensor $\Theta_M{}^\alpha$ has to satisfy a number of consistency conditions. The first constraint is a generalization of (4.49):

$$Q_{MN}{}^\alpha \equiv f_{\beta\gamma}{}^\alpha\Theta_M{}^\beta\Theta_N{}^\gamma + (t_\beta)_N{}^Q\Theta_M{}^\beta\Theta_Q{}^\alpha = 0 , \tag{4.85}$$

where $f_{\alpha\beta}{}^\gamma$ are the structure constants of the global symmetry algebra, $[\delta_\alpha, \delta_\beta] = f_{\alpha\beta}{}^\gamma\delta_\gamma$. It implies that

$$[X_M, X_N] = -X_{MN}{}^P X_P , \tag{4.86}$$

and therefore guarantees the closure of the gauge algebra, with $X_{MN}{}^P$ as its generalized structure constants.

From their definition in (4.81), it is clear that the tensors $X_{MN}{}^P$ are in general not antisymmetric in $[MN]$, and for the moment we will also not impose the condition (4.45) that was required in the case of an electric gauging. However, the symplectic matrices $X_{MN}{}^P$ are not completely arbitrary since the left hand side of (4.86) is antisymmetric in $[MN]$, and therefore, so should be the right hand side. This means we have to impose

$$Y^P{}_{MN}X_P = 0 , \qquad \text{with } Y^P{}_{MN} \equiv X_{(MN)}{}^P . \tag{4.87}$$

Thus the symmetric part of $X_{MN}{}^P$ only vanishes upon contraction with the embedding tensor, but is not zero in itself. This signals a difference with ordinary gauge groups, where the structure constants are antisymmetric and satisfy the Jacobi identity. Writing (4.86) explicitly gives

$$X_{MQ}{}^P X_{NP}{}^R - X_{NQ}{}^P X_{MP}{}^R + X_{MN}{}^P X_{PQ}{}^R = 0 . \tag{4.88}$$

Antisymmetrizing in $[MNQ]$, we can split the second factor of each term into the antisymmetric and symmetric part, $X_{MN}{}^P = X_{[MN]}{}^P + X_{(MN)}{}^P$, and this gives a violation of the Jacobi identity for $X_{[MN]}{}^P$ as

$$\begin{aligned} &X_{[MN]}{}^P X_{[QP]}{}^R + X_{[QM]}{}^P X_{[NP]}{}^R + X_{[NQ]}{}^P X_{[MP]}{}^R \\ &\quad = -\frac{1}{3}\left(X_{[MN]}{}^P X_{(QP)}{}^R + X_{[QM]}{}^P X_{(NP)}{}^R + X_{[NQ]}{}^P X_{(MP)}{}^R\right) . \end{aligned} \tag{4.89}$$

The presence of a non-trivial tensor $Y^M{}_{NP}$ also alters the usual properties of the field strength, which follows from the Ricci identity, $[D_\mu, D_\nu] = -\mathcal{F}_{\mu\nu}{}^M \delta_M$,

$$\mathcal{F}_{\mu\nu}{}^M \equiv 2\partial_{[\mu} A_{\nu]}{}^M + X_{[PQ]}{}^M A_\mu{}^P A_\nu{}^Q \,. \tag{4.90}$$

In particular, it will no longer fulfill the Bianchi identity, which must now be replaced by

$$D_{[\mu}\mathcal{F}_{\nu\rho]}{}^M = Y^M{}_{NP} A_{[\mu}{}^N \mathcal{F}_{\nu\rho]}{}^P - \frac{1}{3} Y^M{}_{PN} X_{[QR]}{}^P \; A_{[\mu}{}^N A_\nu{}^Q A_{\rho]}{}^R \,. \tag{4.91}$$

Furthermore, $\mathcal{F}_{\mu\nu}{}^M$ is not fully covariant under a gauge transformation (4.84). Instead, we have

$$\begin{aligned} \delta(\Lambda)\mathcal{F}_{\mu\nu}{}^M &= 2D_{[\mu}\delta A_{\nu]}{}^M - 2Y^M{}_{PQ} A_{[\mu}{}^P \delta A_{\nu]}{}^Q \\ &= X_{NQ}{}^M \; \mathcal{F}_{\mu\nu}{}^N \Lambda^Q - 2Y^M{}_{PQ} A_{[\mu}{}^P \delta A_{\nu]}{}^Q \,. \end{aligned} \tag{4.92}$$

Both in (4.91) and here, there are non-covariant terms that depend on $Y^M{}_{NP}$.[19]

The violation of the Jacobi identity (4.89), the modified Bianchi identity (4.91), and the non-covariant transformation (4.92) is the prize one has to pay for the symplectically covariant treatment in which both electric and magnetic vector potentials appear at the same time. In order to compensate for these violations and in order to make sure that the number of propagating degrees of freedom is the same as before, one has to make several modifications to the gauge structure that we outlined so far. The character of these changes is inspired by our discussion on magnetic gaugings in §4.3.

The first step towards a solution is the introduction of extra gauge transformations for the vector fields:

$$\delta(\Lambda) A_\mu{}^M \quad \rightarrow \quad \delta(\Lambda, \Xi) A_\mu{}^M = \delta(\Lambda) A_\mu{}^M + \delta(\Xi) A_\mu{}^M \,, \tag{4.93}$$

where

$$\delta(\Xi) A_\mu{}^M \equiv -Y^M{}_{NP} \Xi_\mu{}^{NP}, \tag{4.94}$$

accompanied by new vector-like parameters $\Xi_\mu{}^{NP}(x)$ that are symmetric in the upper indices. The extra $\delta(\Xi)$-transformations contained in (4.93) allow one to gauge away the vector fields that correspond to the directions in which the Jacobi identity is violated, i.e., directions in the kernel of the embedding tensor (see (4.87)).

[19]Due to the vanishing contraction of $Y^M{}_{NP}$ with the embedding tensor (recall (4.87)), it turns out that the field strength contracted with $\Theta_M{}^\alpha$ has in fact all the usual properties. But since one cannot construct a Lagrangian with the tensors $\Theta_M{}^\alpha \mathcal{F}_{\mu\nu}{}^M$, we are interested in the properties of the bare field strength.

It is important to notice that the modified gauge transformations (4.93) still close on the gauge fields and thus form a Lie algebra. Indeed, the commutation relations are

$$\begin{aligned} [\delta(\Lambda_1), \delta(\Lambda_2)]\, A_\mu{}^M &= \delta(\Lambda_3) A_\mu{}^M + \delta(\Xi_3) A_\mu{}^M, \\ [\delta(\Lambda), \delta(\Xi)]\, A_\mu{}^M &= [\delta(\Xi_1), \delta(\Xi_2)]\, A_\mu{}^M = 0\,, \end{aligned} \tag{4.95}$$

with

$$\begin{aligned} \Lambda_3^M &= X_{[NP]}{}^M \Lambda_1^N \Lambda_2^P\,, \\ \Xi_{3\mu}{}^{PN} &= \Lambda_1^{(P} D_\mu \Lambda_2^{N)} - \Lambda_2^{(P} D_\mu \Lambda_1^{N)}\,. \end{aligned} \tag{4.96}$$

Given the new transformations (4.93), we can now also construct new field strengths that are gauge covariant. This is necessary if we want to deform the original Lagrangian (4.57) and accommodate electric and magnetic gauge fields, since $\mathcal{F}_{\mu\nu}{}^M$ cannot be used to construct gauge-covariant kinetic terms. For this purpose, we introduce tensor fields $B_{\mu\nu}{}^{NP}$, symmetric in $(NP)$, and with them new field strengths

$$\mathcal{H}_{\mu\nu}{}^M \equiv \mathcal{F}_{\mu\nu}{}^M + Y^M{}_{NP} B_{\mu\nu}{}^{NP}\,. \tag{4.97}$$

The gauge transformations of $B_{\mu\nu}{}^{NP}$ are fixed by demanding the covariant transformation of $\mathcal{H}_{\mu\nu}{}^M$. More explicitly, we determine $\delta(\Lambda, \Xi) B_{\mu\nu}{}^{NP}$ such that

$$\delta(\Lambda, \Xi)\mathcal{H}_{\mu\nu}{}^M = -\Lambda^P X_{PN}{}^M \mathcal{H}_{\mu\nu}{}^N\,. \tag{4.98}$$

We find

$$\delta(\Lambda, \Xi) B_{\mu\nu}{}^{NP} = 2D_{[\mu}\Xi_{\nu]}{}^{NP} + 2A_{[\mu}{}^{(N}\delta A_{\nu]}{}^{P)} - 2\Lambda^{(N}\mathcal{H}_{\mu\nu}{}^{P)}\,, \tag{4.99}$$

with

$$D_\mu \Xi_\nu{}^{NP} = \partial_\mu \Xi_\nu{}^{NP} + X_{QR}{}^P A_\mu{}^Q \Xi_\nu{}^{NR} + X_{QR}{}^N A_\mu{}^Q \Xi_\nu{}^{PR}\,. \tag{4.100}$$

Note that the transformation (4.99) can always be supplemented by extra terms that vanish upon contraction with $Y^M{}_{NP}$. Since the 2-forms will be contracted with $Y^M{}_{NP}$ in the remainder of this chapter, we will not consider these extra terms here.

### Gauge invariant action in $D = 4$

We have now introduced the minimal amount of ingredients that led to the construction of a consistent gauge algebra and covariant field strengths. In addition to the vector fields and the $\delta(\Lambda)$-transformations, consistency required the introduction of additional local transformations and new 2-form tensor fields. So far, we have not introduced any dynamics for the fields, or made reference to an action. However, the discussion in the previous section can be embedded into a framework where such an action is present. A general expression in $D = 4$ was first given in [61] and contains kinetic and Chern-Simons terms for the vectors, topological terms for the 2-forms and general matter terms:

$$\mathcal{L}_{\rm VT} = \mathcal{L}_{\rm g.k.} + \mathcal{L}_{\rm GCS} + \mathcal{L}_{{\rm top},B} + \mathcal{L}_{\rm matter}\,, \tag{4.101}$$

with

$$\begin{aligned}
\mathcal{L}_{\rm g.k.} &= \frac{1}{4} e \mathcal{I}_{\Lambda\Sigma} \mathcal{H}_{\mu\nu}{}^{\Lambda} \mathcal{H}^{\mu\nu\Sigma} - \frac{1}{8} \mathcal{R}_{\Lambda\Sigma} \varepsilon^{\mu\nu\rho\sigma} \mathcal{H}_{\mu\nu}{}^{\Lambda} \mathcal{H}_{\rho\sigma}{}^{\Sigma}\,, && (4.102)\\
\mathcal{L}_{\rm GCS} &= \varepsilon^{\mu\nu\rho\sigma} A_{\mu}{}^{M} A_{\nu}{}^{N} \left( \frac{1}{3} X_{MN\,\Lambda}\, \partial_{\rho} A_{\sigma}{}^{\Lambda} + \frac{1}{6} X_{MN}{}^{\Lambda} \partial_{\rho} A_{\sigma\Lambda} \right.\\
&\qquad \left. + \frac{1}{8} X_{MN\,\Lambda} X_{PQ}{}^{\Lambda} A_{\rho}{}^{P} A_{\sigma}{}^{Q} \right)\,, && (4.103)\\
\mathcal{L}_{{\rm top},B} &= \frac{1}{4} \varepsilon^{\mu\nu\rho\sigma}\, Y^{\Lambda}{}_{NP}\, B_{\mu\nu}{}^{NP} \left( \mathcal{F}_{\rho\sigma\,\Lambda} + \frac{1}{2} Y_{\Lambda RS}\, B_{\rho\sigma}{}^{RS} \right)\,. && (4.104)
\end{aligned}$$

The tensors $\mathcal{I}_{\Lambda\Sigma}$ and $\mathcal{R}_{\Lambda\Sigma}$ are the real and imaginary part of the scalar dependent gauge kinetic function. In [2] it was pointed out that the Lagrangian (4.101) is not automatically gauge invariant under the transformations (4.93) and (4.99). Indeed, the structure that we have outlined so far, has to be supplemented by two extra ingredients.

(i) The embedding tensor has to satisfy an additional constraint, known in the literature as the linear or representation constraint. It has the following form:

$$D_{MNP} \equiv X_{(MN}{}^{Q} \Omega_{P)Q} = 0\,. \tag{4.105}$$

This constraint was first found as a necessary condition for supersymmetry invariance of theories with a maximal amount of supercharges [65, 70, 83]. However, it also plays a crucial role in showing gauge invariance. In chapter 6, a physical interpretation will be given to this constraint.

(ii) The second modification concerns the transformations of the 2-forms. They need to be supplemented by extra terms that reflect the dependence of the Lagrangian on the matter content. So once we specify the dynamics for the fields, the transformations of the 2-forms are

$$\delta(\Lambda,\Xi) B_{\mu\nu}{}^{NP} = 2 D_{[\mu} \Xi_{\nu]}{}^{NP} + 2 A_{[\mu}{}^{(N} \delta A_{\nu]}{}^{P)} - 2 \Lambda^{(N} \mathcal{H}_{\mu\nu}{}^{P)} + \Delta B_{\mu\nu}{}^{NP}\,, \tag{4.106}$$

with

$$\Delta B_{\mu\nu}{}^{NP} = -2\Lambda^{(N}\left(\mathcal{G}_{\mu\nu}{}^{P)} - \mathcal{H}_{\mu\nu}{}^{P)}\right), \tag{4.107}$$

which depends on the scalar fields via the dual field strengths

$$\mathcal{G}_{\mu\nu}{}^{M} = (\mathcal{H}_{\mu\nu}{}^{\Lambda}, \mathcal{G}_{\mu\nu\Lambda}), \tag{4.108}$$

$$\text{with} \quad \mathcal{G}_{\mu\nu\,\Lambda} \equiv \varepsilon_{\mu\nu\rho\sigma}\frac{\partial\mathcal{L}}{\partial\mathcal{H}_{\rho\sigma}{}^{\Lambda}} = \mathcal{R}_{\Lambda\Gamma}\mathcal{H}_{\mu\nu}{}^{\Gamma} + \frac{1}{2}e\,\varepsilon_{\mu\nu\rho\sigma}\,\mathcal{I}_{\Lambda\Gamma}\,\mathcal{H}^{\rho\sigma\,\Gamma}\,.$$

To summarize, the action (4.101) is invariant under the new set of gauge transformations (4.93) and (4.106), provided that we use the closure and linear constraints on the embedding tensor. For a proof of this result we refer the reader to §6.2.

Before we conclude this section with a short summary and outlook, let us still make a few important remarks about the structure and the properties of the embedding tensor formalism.

## Conclusions

**1.** All the relevant formulae acquire a universal form in terms of the spurious embedding tensor. Once a particular choice for $\Theta_M{}^\alpha$ is made, its entries are constant parameters that determine the gauging completely. In particular, one has to evaluate the Lagrangian (4.101) and the gauge transformations for the particular choice of $\Theta_M{}^\alpha$.

**2.** The embedding tensor is not an arbitrary matrix, but it should satisfy the closure and representation constraints. Obviously one can impose additional constraints (an example will be given in point 4. on the next page), but the above set is certainly the minimal one. For a given global symmetry group and a given vector field content of the ungauged theory, one can solve these constraints and obtain all admissible embedding tensors. In this way, we obtain a complete classification of all gauged deformations of a given ungauged theory.

**3.** Formally, the embedding tensor formalism preserves electromagnetic duality covariance on-shell. Indeed, the equations of motion are

$$\frac{\partial S}{\partial A_\mu{}^M} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\Omega_{MN}D_\nu\mathcal{G}_{\rho\sigma}{}^N - J^\mu{}_M \approx 0\,, \tag{4.109}$$

$$\frac{\partial S}{\partial B_{\mu\nu}{}^{MN}} = \frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\Omega_{RS}Y^R{}_{MN}\,(\mathcal{H}-\mathcal{G})_{\rho\sigma}{}^S \approx 0\,, \tag{4.110}$$

with $J^\mu{}_M \equiv \frac{\partial S_{\text{matter}}}{\partial A_\mu{}^M}$ and identifications on-shell are denoted by $\approx$. The first equation (4.109) is the generalization of (4.60) and has clearly a symplectic covariant form. The second relation (4.110) guarantees that the magnetic field strengths $\mathcal{H}_{\mu\nu\Lambda}$ are related to the electric field strengths via (4.108), at least for the components projected by $\Theta^{\Lambda\alpha}$.

Equation (4.110) and the magnetic part of (4.109) do not play the role of dynamical field equations, but together with the combined vector and tensor gauge invariances they ensure that the number of propagating degrees of freedom has not changed upon the introduction of tensor and magnetic vector fields in the gauged theory (see e.g. §5.1 in [61]). In close analogy to our examples in §4.3, it is possible to integrate out the 2-forms and gauge fix the remaining tensor gauge transformations. This yields a Lagrangian that contains precisely $n_V$ physical vector fields. This Lagrangian does not necessarily describe a continuous deformation of the ungauged Lagrangian, but will always be related to it via an electromagnetic duality transformation. This might sound as if we have not achieved anything new at all by considering magnetic gaugings, which is not exactly true. First of all we stress again that a classification of all gaugings is much easier if one does not have to scan all duality frames by hand for "gaugeable" subgroups of the invariance group. Moreover, it has become apparent that unusual duality frames (i.e., those involving magnetic vectors and 2-forms) are generically present in superstring and M-theory compactifications, such as dimensional reductions with fluxes or generalized Scherk-Schwarz reductions. It is in this context that the embedding tensor formalism becomes very useful.

**4.** Finally, we still want to mention a technical issue that pops up at several places in the literature, but which is not so important for the remainder of our discussion. From the representation and linear constraints it follows that also the following restriction on the embedding tensor is satisfied:

$$E^{\tilde{\alpha}\tilde{\beta}} = \Omega^{MN}\Theta_M{}^{\tilde{\alpha}}\Theta_N{}^{\tilde{\beta}} = 0\,, \tag{4.111}$$

where the tilded indices $\tilde{\alpha},\ldots$ form a subset of the adjoint indices $\alpha,\ldots$, such that $(t_{\tilde{\alpha}})_M{}^N \neq 0$. The extension of (4.111) to all adjoint indices, i.e. $E^{\alpha\beta} = 0$, is called the locality constraint, and it guarantees that all electric and magnetic charges that appear in the gauging are mutually local. Equivalently, one can always make a duality transformation such that all charges in the gauged theory are converted to electric ones. As we pointed out already, the locality constraint will not play a role in the remainder of this text.

## 4.4 Summary and outlook

Let us recapitulate the most important results of this section. We started with a discussion about the global internal symmetries of $4D$ basic supergravity theories. These symmetries are fixed to be isometries of the scalar manifold, in combination with electromagnetic duality transformations of the vector field strengths and their duals. In general, the latter do not leave the action invariant because the action depends on the chosen duality frame. Once the global symmetries are known, it is a short step to consider the effects of adding a non-trivial gauge parameter to the basic supergravities. In a first attempt we had to restrict to the gauging of symmetries that are contained in the invariance group of the action. These are the so-called conventional (or electric) gaugings, which are based on the addition of minimal couplings to the electric vector fields. Although we are very familiar with this procedure in physics, the downside of this construction is that it breaks on-shell electromagnetic covariance explicitly, and that it does not allow a systematic study of all possible deformations. This issue was solved in the final part of this chapter, where we introduced the embedding tensor formalism. This framework formally restores on-shell electromagnetic covariance, at the cost of introducing magnetic gauge fields, antisymmetric 2-forms and extra (vector-like) gauge generators. On the up-side, a group-theoretical analysis of the embedding tensor and its constraints leads to a complete classification of all possible deformations with a non-vanishing gauge parameter.

With all this information at hand, we are finally ready to start the presentation of our research results. The discussion will be spread over three chapters that roughly correspond to the publications [1], [2] and [3].

- In *chapter 5* we discuss the presence of gauge and gravitational anomalies in theories with $\mathcal{N} = 1$ global or local supersymmetry and a conventional gauging. We present a Green-Schwarz mechanism that involves Peccei-Quinn terms, generalized Chern-Simons terms, higher order derivative corrections and appropriate gauge transformations of the scalar fields. We discuss the mutual consistency conditions for all these ingredients, such that the theory is anomaly-free.

- In *chapter 6* we will extend our results about gauge anomaly cancellation to the framework of generalized gaugings and the embedding tensor formalism. We will see that anomaly cancellation requires the modification of the representation constraint (4.105), and we will interpret the original constraint as the condition for anomaly freedom.

- Finally, in *chapter 7* we study the structure of the embedding tensor formalism (in the absence of quantum anomalies) in more detail. In particular, we compare the gauge transformations dependent and independent of an invariant action (i.e., $\Delta B_{\mu\nu}{}^{MN} \neq 0$ and $\Delta B_{\mu\nu}{}^{MN} = 0$ respectively), and argue that the generic transformations lead to an infinitely reducible algebra. We connect the embedding tensor formalism to the field-antifield (or Batalin-Vilkovisky) formalism, which is the most general formulation known for general gauge theories and their quantization.

CHAPTER 5

# ANOMALY CANCELLATION IN $D = 4, \mathcal{N} = 1$ SUPERGRAVITY

*The chapter at hand contains an overview of the research results that were obtained in collaboration with Jan Rosseel, Torsten T. Schmidt, Antoine Van Proeyen and Marco Zagermann, and were published in [1]. We clarify the interplay of Peccei-Quinn terms, generalized Chern-Simons terms and quantum gauge anomalies in the context of $\mathcal{N} = 1$ supergravity and exhibit conditions that have to be satisfied for their mutual consistency. We also present some unpublished results on a Green-Schwarz mechanism for mixed gauge-gravitational anomalies and clarify the role of higher order derivative corrections to the original (2-derivative) supergravity Lagrangian. Our results provide the supersymmetric framework for studies of string compactifications with axionic shift symmetries, generalized Chern-Simons terms and quantum anomalies.*

## 5.1 Introduction

The context of our work consists of the 4-dimensional gauge theories[1] with a minimal amount of global or local supersymmetry. The generic structure of these

[1] We only consider the conventional gaugings that were discussed in §4.2. An extension of our results to generalized gaugings and the embedding tensor formalism will be presented in *chapter 6*.

theories was discussed in the previous chapter. It turns out that, in general, they have a chiral field content and couplings that are different for left and right handed fermions. In particular, there can be parity-violating gauge couplings. As such, these theories might suffer from local anomalies, as we pointed out in §2.4, and are potentially ill-defined at the quantum level. A detailed analysis reveals that the following three types of anomalies can occur:[2]

- *Gauge anomalies.* These were discussed to some detail in §2.4, and they manifest themselves as a non-invariance of the effective action under gauge transformations, see (2.85).
- *Supersymmetry anomalies.* The compatibility between gauge anomalies and supersymmetry usually triggers a violation of the latter. This is best seen from the modified supersymmetry algebra in (4.59). If the effective action is not invariant under gauge transformations, it cannot be invariant under supersymmetry either.
- *Mixed gauge-gravitational anomalies.* Whereas gauge and supersymmetry anomalies occur in theories with global or local supersymmetry, mixed anomalies are specific for gauged supergravities. They manifest themselves as a non-invariance of the effective action under local Lorentz transformations. Alternatively, they can be computed from triangle one-loop diagrams with chiral fermions running in the loop and two energy momentum tensors and a $U(1)$ gauge current at the external legs.

The main purpose of our work is to identify the necessary and sufficient conditions, such that generic $\mathcal{N} = 1$ theories with chiral matter couplings are free from the above anomalies.

In §2.4 we saw that the requirement of anomaly-freedom imposes a number of nontrivial constraints on the possible gauge quantum numbers of the chiral fermions. The strongest requirements are obtained if one demands that all anomalous one-loop diagrams due to chiral fermions simply add up to zero. However, these constraints on the fermionic spectrum can be somewhat relaxed if some of the anomalous one-loop contributions are instead canceled by *classical* gauge-variances of certain terms in the tree-level action. The prime example for this is the Green-Schwarz (GS) mechanism [25].

In the first part of our discussion, we present an exhaustive analysis of the four-dimensional incarnation of the GS mechanism for theories with *rigid*

[2] Remark that there are no pure gravitational anomalies in 4 dimensions. The 4-dimensional $CPT$ theorem guarantees that for every fermion with one helicity, there is an antiparticle with the same mass but opposite helicity. Since the gravitational interactions of a fermion only depend on the mass and helicity, the overall gravitational coupling cannot break parity.
Moreover, we did also not mention (mixed) Kähler anomalies which are manifestly present if local $U(1)_R$ symmetry is broken. We refer to the literature [87–91] for more details.

supersymmetry. We will see that it uses the gauge variance of Peccei-Quinn terms (recall (4.58)) to cancel (part of) the anomaly. However, it has recently been pointed out that this is often not sufficient to cancel all quantum anomalies. Indeed, in the context of particular orientifold compactifications with intersecting D-branes and an anomalous fermion spectrum [10, 11, 92], it has been shown that the cancellation of certain mixed Abelian anomalies needs an additional ingredient in the classical action. This extra ingredient is the so-called generalized Chern-Simons (GCS) term, which has two parts that are of the schematic form $A \wedge A \wedge dA$ and $A \wedge A \wedge A \wedge A$, where the vector fields $A$ are not all the same. It is quite obvious that these terms are not gauge invariant, and it is precisely this gauge variance that can be used in some cases to cancel possible left-over gauge variances from quantum anomalies and Peccei-Quinn terms.

Besides the presence of GCS terms in the context of orientifold compactifications and an anomalous fermion spectrum, they have also appeared in various works where no reference to anomalies is made. For example, in [12] various higher-dimensional origins of GCS terms are described, such as certain flux and generalized Scherk-Schwarz compactifications. In each of these compactifications, GCS terms are required to show gauge invariance of the resulting 4-dimensional classical action. Moreover, GCS terms lead to interesting phenomenological signatures for certain variants of $Z'$-bosons [11, 92], and they play an important role in the manifestly symplectic formulation of generalized gaugings introduced in §4.3.

In view of these applications, it is surprising that the full interplay between GCS terms, gauge invariance and quantum anomalies has never been investigated in the context of minimal supersymmetry. In fact, before the work of [12], supersymmetric GCS terms were only studied in the context of extended supersymmetry [41, 65, 70, 77, 80, 83].[3] The primary goal of our work is to close this gap and clarify how the ingredients (each of which individually breaks gauge symmetry)

(i) quantum gauge anomalies,

(ii) Peccei-Quinn terms, and

(iii) generalized Chern-Simons terms

can be compatible with global $\mathcal{N} = 1$ supersymmetry.

The second part of this chapter deals with the issue of anomaly cancellation in $\mathcal{N} = 1$ *supergravity*. An extension of the GS mechanism for gauge and supersymmetry anomalies turns out to be rather straightforward. On the other

[3] We should note that the context of extended supersymmetry is qualitatively different since there are no chiral gauge couplings and hence no quantum anomalies.

hand, the presence of gravity also raises the question of possible mixed gauge-gravitational anomalies. In earlier works [93–96] a beginning has been made to the study of appropriate counterterms that are needed to cancel these mixed anomalies. Most of the discussion in these papers was motivated by a reduction of the 10-dimensional GS mechanism for heterotic strings to 4 dimensions. This reduction provides evidence for the need of higher order corrections to the standard supergravity action. In particular, one finds a coupling of the axion field to the so-called Hirzebruck signature density, which is quadratic in the Riemann tensor. This term is not invariant under appropriate (shift) transformations of the axion, and under certain conditions, it cancels the mixed anomaly. In §5.7 we will review the construction of the higher derivative counterterms (including their supersymmetric completion), and we study the exact conditions that are necessary for mixed anomaly cancellation. We should point out to the reader that the details have not been fully worked out yet, and only the first steps towards a solution are being presented. In the future, more work needs to be done and our partial results can be used as a starting point.

The outline of this chapter is as follows. In §5.2 we present a short review of the GS mechanism in its original ten-dimensional form, as it was discovered by Green and Schwarz in [25]. There are several good reasons for doing this. In the first place, the $10D$ mechanism is well understood and its transparent structure gives the reader some insight into the $4D$ mechanism, which is very similar. But most importantly, the four-dimensional GS mechanism is related to its ten-dimensional counterpart via dimensional reduction. We will present the results of a simple Calabi-Yau compactification, which leads to $4D$ minimal supergravity with an Abelian gauge group and possibly anomalous chiral fermions. This easy example will allow us to identify the crucial ingredients for anomaly cancellation in superstring inspired $4D$, $\mathcal{N} = 1$ supergravity.[4] Subsequently, we will expand on this result and develop the general four-dimensional GS mechanism in a fully supersymmetric setting. This discussion involves several parts; in §5.3 we provide more details about the structure and transformation rules of $\mathcal{N} = 1$ globally supersymmetric theories with a local gauge group. In particular, we will introduce the generalized Chern-Simons that are crucial for anomaly cancellation, as we pointed out before. In §5.4, we consider the quantum gauge and supersymmetry anomalies as obtained from the variation of the effective action in [97, 98], and we analyze the complete cancellation of these anomalies by using the results of the previous section. To show how this works in practice, it is useful to look at a gauge group that is the product of an Abelian and a semi-simple group; this will be done in §5.5. The last two sections are devoted to a discussion about the supergravity corrections.

[4]This simple reduction does not give rise to GCS terms, though. Nevertheless, it is sufficient to illustrate the basic principles of 4-dimensional anomaly cancellation. As we discussed above, GCS terms and a more general cancellation mechanism can be obtained via more general reductions such as compactifications on orientifolds.

Section 5.6 contains an extension of the results from §5.4, and §5.7 discusses the Green-Schwarz mechanism for mixed gauge-gravitational anomalies.

## 5.2 The Green-Schwarz mechanism

**The original mechanism in $10D$**

The $10D$ gravity multiplet contains massless fermions of one chirality but not the other. Moreover, the $CPT$ operation in 10 dimensions leaves invariant the chirality of a massless fermion, contrary to $4D$ (see footnote 2). Therefore, it is possible to construct parity violating interactions in $10D$ supergravity. These chiral couplings can give rise to gauge, pure gravitational and mixed anomalies. The general form of these anomalies has been characterized in [99] (see also [100, 101] for simultaneous developments), and was used to single out the $10D$ supergravities that are anomaly-free. The results are as follows:

**Type IIA supergravity.** This theory is non-chiral and therefore it has no anomalies.

**Type IIB supergravity.** A remarkable cancellation occurs between the anomaly contributions of the chiral gravitino, a complex spinor with opposite chirality, and a self-dual antisymmetric tensor field strength [99].

**Type I and heterotic supergravity.** There is no immediate cancellation as in the type IIB case. Instead, a detailed calculation of the anomaly reveals a complicated non-vanishing result. Nevertheless, Green and Schwarz discovered a way to cancel the anomaly in the cases where it has the following special form [25]:

$$\mathcal{A} \equiv \delta\Gamma = \left(\frac{2}{3}+\alpha\right)\int\left(\omega_3^{(\mathrm{L})}-\omega_3^{(\mathrm{YM})}\right)dX_{6,1}+\left(\frac{1}{3}-\alpha\right)\int\left(\omega_{2,1}^{(\mathrm{L})}-\omega_{2,1}^{(\mathrm{YM})}\right)X_8\,, \tag{5.1}$$

where $\alpha$ is an arbitrary parameter and we used the notation $X_{m,n}$ to indicate a $m$-form that contains $n$ gauge parameters. In particular, $dX_{6,1}$ is given by the combined local Lorentz and gauge variations of a seven-form: $\delta(\Lambda,\lambda)X_7 = dX_{6,1}$, and $X_8$ is some eight-form. Moreover, the anomaly contains the Lorentz and Yang-Mills Chern-Simons three-forms $\omega_3^{(\mathrm{L})}$ and $\omega_3^{(\mathrm{YM})}$, which are defined as follows:

$$\omega_3^{(\mathrm{L})} = \mathrm{tr}(\omega d\omega+\frac{2}{3}\omega^3)\,, \tag{5.2}$$

$$\omega_3^{(\mathrm{YM})} = \mathrm{Tr}(AdA+\frac{2}{3}A^3)\,, \tag{5.3}$$

where we used tr and Tr to denote a trace in the vector and adjoint representation respectively. In the last term of (5.1), certain two-forms $\omega_{2,1}^{(\mathrm{L})}$ and $\omega_{2,1}^{(\mathrm{YM})}$ appear that are related to the local Lorentz and gauge transformation of $\omega_3^{(\mathrm{L})}$ and $\omega_3^{(\mathrm{YM})}$ respectively:

$$\delta(\lambda)\omega_3^{(\mathrm{L})} \equiv d\omega_{2,1}^{(\mathrm{L})} \quad \rightarrow \quad \omega_{2,1}^{(\mathrm{L})} = \mathrm{tr}(\lambda d\omega)\,, \tag{5.4}$$

$$\delta(\Lambda)\omega_3^{(\mathrm{YM})} \equiv d\omega_{2,1}^{(\mathrm{YM})} \quad \rightarrow \quad \omega_{2,1}^{(\mathrm{YM})} = \mathrm{Tr}(\Lambda dA)\,. \tag{5.5}$$

It turns out that the anomaly $\mathcal{A}$ only takes the special form in (5.1) for a limited number of theories. These are the type I supergravities with a $SO(32)$ or $E_8 \times E_8$ gauge group [25], and heterotic supergravity which has a $SO(16) \times SO(16)$ gauge group [102, 103]. Then Green and Schwarz realized that the anomaly $\mathcal{A}$ can be canceled if one adds the following local counterterms to the effective action $\Gamma$:

$$\Delta\Gamma = \int BX_8 - \left(\frac{2}{3} + \alpha\right) \int \left(\omega_3^{(\mathrm{L})} - \omega_3^{(\mathrm{YM})}\right) X_7\,, \tag{5.6}$$

where $B$ is the NSNS 2-form. An appropriate transformation of this two-form, namely

$$\delta(\Lambda,\lambda)B = \omega_{2,1}^{(\mathrm{YM})} - \omega_{2,1}^{(\mathrm{L})}\,, \tag{5.7}$$

leads to a vanishing local Lorentz and gauge variation of $\Gamma + \Delta\Gamma$:

$$\delta(\Lambda,\lambda)\left(\Gamma + \Delta\Gamma\right) = \mathcal{A} + \delta(\Lambda,\lambda)\Delta\Gamma = 0\,. \tag{5.8}$$

Therefore, we say that the counterterms $\Delta\Gamma$ cancel the anomaly. This is the Green-Schwarz mechanism.

The reader might wonder what is the origin of the counterterms $\Delta\Gamma$. The stringy interpretation is that they arise from integrating out some massive string modes from one-loop diagrams. Therefore, they carry an explicit factor of $\hbar$. This makes them eligible candidates to cancel the anomaly, which is a one-loop effect of massless particles at string tree level, and therefore $\mathcal{A}$ is also proportional to $\hbar$.

Finally, we remark that the two-forms should appear in the action with a kinetic term that is invariant under the transformations (5.7). Therefore, we define the field strength

$$\tilde{H} = dB - \omega_3^{(\mathrm{YM})} + \omega_3^{(\mathrm{L})}\,, \tag{5.9}$$

which generalizes the original definition $H = dB - \omega_3^{(\mathrm{YM})}$.[5] The kinetic term is then proportional to $\tilde{H}^2$, and it involves terms that are linear and quadratic in $\omega_3^{(\mathrm{L})}$.

[5] The Yang-Mills Chern-Simons form $\omega_3^{(\mathrm{YM})}$ is already necessary to write down a consistent supersymmetric coupling between non-Abelian vector multiplets and gravity [104, 105], but plays also a crucial role in the GS mechanism.

These terms are fourth and sixth order in the space-time derivatives respectively, and therefore they are not present in the original truncation of string theory to zeroth order in $\alpha'$. Moreover, the supersymmetric completion of these higher order terms has not been constructed so far.

To summarize, we have seen that the ten-dimensional GS mechanism requires the subtle interplay between several ingredients. The original type I and heterotic supergravity effective actions need to be modified with additional counterterms $\Delta\Gamma$ and higher order terms depending on $\omega_3^{(\mathrm{L})}$. The classical variation of these extra contributions cancels the quantum anomaly, given the appropriate transformation of the NSNS two-forms $B$. Since this result holds exactly, it must also hold in any valid approximation such as the reduction to a four-dimensional effective theory after compactification. In the remainder of this section we will sketch the outlines of how this works. In §5.2 we will determine the relevant parts in the reduction of the (modified) $10D$ effective action to $4D$, and in §5.2 we discuss the appropriate field transformations that lead to a cancellation of the $4D$ anomaly. Again, we remind the reader that this discussion only summarizes the general structure and relevant ingredients of the four-dimensional GS mechanism as it follows from its ten-dimensional equivalent. A detailed and independent treatment will follow in sections 5.3 through 5.7.

### Reduction to 4 dimensions

The CY compactification –in a particularly manageable case– of ten-dimensional heterotic superstring model to $4D$, $\mathcal{N}=1$ theories was first considered by Witten in [106, 107]. He studied the effect of the field strength $\tilde{H}$, although without the higher derivative Lorentz Chern-Simons term $\omega_3^{(\mathrm{L})}$. The result is a coupling of the imaginary part of a scalar field $z$ (the axion) to a topological term depending on the vector fields:

$$\mathcal{L}_{\mathrm{PQ}} = \frac{1}{8}\mathrm{Im}z\,\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\,. \tag{5.10}$$

We recognize this as the usual Peccei-Quinn (PQ) Lagrangian. Apart from the PQ term, also the standard kinetic term,

$$e^{-1}\mathcal{L}_{\mathrm{kin}} = -\frac{1}{4}\mathrm{Re}z\,F_{\mu\nu}F^{\mu\nu}, \tag{5.11}$$

is found through dimensional reduction. In the light of further developments it is useful to combine $\mathcal{L}_{\mathrm{PQ}}$ and $\mathcal{L}_{\mathrm{kin}}$ in the following (manifestly supersymmetric) superfield expression:

$$\mathcal{L}_{\mathrm{g.k.}} = \int \mathrm{d}^2\theta\;\mathcal{E}(S\,W_\alpha W_\beta\,\varepsilon^{\alpha\beta}) + \mathrm{h.c.}, \tag{5.12}$$

where $S$ is a chiral superfield with $z$ as its lowest component and $W_\alpha$ is the supersymmetric extension of the field strength. The latter is defined in terms of a real superfield $V$ as follows: $W_\alpha = \frac{1}{4}(\bar{\mathcal{D}}^2 - 8R)\mathcal{D}_\alpha V$, or a generalization thereof for the non-Abelian case. As a consequence of this definition, $\mathcal{L}_{\text{g.k.}}$ can also be written as a $D$-term [12, 95]:

$$\mathcal{L}_{\text{g.k.}} = \int \mathrm{d}^4\theta \, \left(S + \bar{S}\right) \, \Omega^{(\text{YM})} \,, \tag{5.13}$$

where we introduced the Yang-Mills Chern-Simons form

$$W_\alpha W_\beta \varepsilon^{\alpha\beta} = \left(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R\right) \Omega^{(\text{YM})} \,, \tag{5.14}$$

$$\Omega^{(\text{YM})} = (\mathcal{D}^\alpha V) W_\alpha + (\bar{\mathcal{D}}_{\dot\alpha} V)\bar{W}^{\dot\alpha} + V\mathcal{D}^\alpha W_\alpha \,. \tag{5.15}$$

The superfield $\Omega^{(\text{YM})}$ has a $\theta$-expansion with the Chern-Simons 3-form $\Omega^{(\text{YM})}_{\mu\nu\rho}$, defined in (5.5), as its $\theta\bar\theta$-component.

The result of Witten in (5.12) can be extended to include the Lorentz Chern-Simons part of the modified field strength $\tilde{H}$. Then the 4-dimensional Lagrangian becomes [93, 94, 108]

$$\begin{aligned} e^{-1}\mathcal{L}_{4D} \;\; &= \;\; -\frac{1}{4}\text{Re}z \, F_{\mu\nu}F^{\mu\nu} \\ &\quad + \frac{1}{8}e^{-1}\text{Im}z \, \varepsilon^{\mu\nu\rho\sigma}\left(F_{\mu\nu}F_{\rho\sigma} - 2R_{\mu\nu ab}R_{\rho\sigma}{}^{ab}\right) + \ldots \,. \end{aligned} \tag{5.16}$$

The last term –quadratic in the Ricci tensor and also called the Hirzebruck signature density– is obviously not present in the original class of $4D$, $\mathcal{N}=1$ supergravity Lagrangians (recall equation (4.57)) since that analysis was restricted to no more than 2 derivatives. Here we see that anomaly cancellation, inspired by the $10D$ GS mechanism, forces us to introduce higher order corrections.

Similar to (5.12), the supersymmetric completion of $\text{Im}z \, \varepsilon^{\mu\nu\rho\sigma}R_{\mu\nu ab}R_{\rho\sigma}{}^{ab}$ can be obtained from a superfield expression:

$$\mathcal{L}_{\text{Weyl}} = -\int \mathrm{d}^2\theta \, \mathcal{E}\left(SW_{\mu\nu\alpha}W^{\mu\nu}_\beta \varepsilon^{\alpha\beta}\right) + \text{h.c.} \,. \tag{5.17}$$

Here we have introduced the (chiral) Weyl multiplet, $W_{\mu\nu\alpha}$, that contains the ordinary Weyl tensor from general relativity in its $\theta$-component. For completeness, we note that $\mathcal{L}_{\text{Weyl}}$ can also be written as a $D$-term, if we introduce the Lorentz Chern-Simons form $\Omega^{(\text{L})}$:

$$\mathcal{L}_{\text{Weyl}} = -\int \mathrm{d}^4\theta \, \left(S + \bar{S}\right) \, \Omega^{(\text{L})} \,, \tag{5.18}$$

$$W_{\mu\nu\alpha}W^{\mu\nu}_\beta \varepsilon^{\alpha\beta} = \left(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R\right) \Omega^{(\text{L})} \,. \tag{5.19}$$

It is known that a superfield expression for $\Omega^{(\mathrm{L})}$ should exist [95], but it has never been constructed explicitly. Also unlike the Yang-Mills case, $\Omega^{(\mathrm{L})}$ is only defined up to additional terms of the form $\frac{1}{2}E_{\alpha\dot{\alpha}}E^{\alpha\dot{\alpha}} - 2RR^*$, with $E_{\alpha\dot{\alpha}}$ the Ricci multiplet that contains the Ricci tensor in its highest component, and $R$ the (chiral) scalar curvature multiplet which is the supersymmetric generalization of the ordinary curvature scalar $R$. Both terms lead to extra higher order contributions that are necessary to avoid the propagation of new massive particles with negative norm state [109] (this is the super-Gauss-Bonnet theorem). However, we will not further consider these extra contributions.

We conclude that the supersymmetric completion of Wittens Lagrangian [107], plus the Lorentz Chern-Simons interactions is

$$\mathcal{L}_{4D} = \int \mathrm{d}^4\theta \, \left(S + \bar{S}\right) \left(\Omega^{(\mathrm{YM})} - \Omega^{(\mathrm{L})}\right) . \tag{5.20}$$

This four-dimensional expression incorporates the remnants of the ten-dimensional GS mechanism in a fully supersymmetric way. In the next section we will see how an appropriate transformation of the superfield $S$ leads to an exact cancellation of the four-dimensional anomaly (under certain conditions).

### The $4D$ GS mechanism: basic facts

We assume a shift transformation of the scalar field under the Abelian gauge group:

$$\delta(\Lambda)S = m\,\Lambda, \qquad \delta(\Lambda)V = \Lambda + \bar{\Lambda}, \tag{5.21}$$

or in components:

$$\begin{aligned} \delta(\Lambda)z &= \mathrm{i}m\Lambda(x) \quad , \qquad & \delta(\Lambda)\chi &= 0 \, , \\ \delta(\Lambda)V_\mu &= \partial_\mu\Lambda(x) \quad , \qquad & \delta(\Lambda)\lambda &= 0 \, , \end{aligned} \tag{5.22}$$

for a real constant $m$ and a real superfield $\Lambda$. The local parameter $\Lambda(x)$ is the highest components of the superfield $\Lambda$. The gauge kinetic function, $f_{\Lambda\Sigma}(z)$, is given by $f_{11}(z) = z$, where the indices $\Lambda, \Sigma, \ldots$ take only one value. Its transformation under the Abelian gauge group is induced by the transformation of the scalar field:

$$\delta(\Lambda)f_{11}(z) = \mathrm{i}m\Lambda(x) = \mathrm{i}\Lambda(x)X_{111} \, , \tag{5.23}$$

where we used the notation from (4.48) in the last equality. Since all structure constants ${f_{\Lambda\Sigma}}^{\Omega}$ are zero, the general expression (4.48) reduces to a shift transformation with a totally symmetric tensor $X_{111} = X_{(111)}$. In (4.58) we saw that the presence of such a shift transformation is problematic for the gauge invariance of the action. Indeed, we find the following non-vanishing variation of

$\mathcal{L}_{4D}$ under the transformations in (5.21):

$$\begin{aligned}\delta(\Lambda)\mathcal{L}_{4D} &= \delta(\Lambda)\int \mathrm{d}^4\theta\ (S+\bar{S})\left(\Omega^{(\mathrm{YM})}-\Omega^{(\mathrm{L})}\right) \\ &= \int \mathrm{d}^4\theta\ X_{111}\left(\Lambda+\bar{\Lambda}\right)\left(\Omega^{(\mathrm{YM})}-\Omega^{(\mathrm{L})}\right). \end{aligned} \tag{5.24}$$

Let us now study each of the non-zero contributions to $\delta(\Lambda)\mathcal{L}_{4D}$ in some detail.

**1.** In order to obtain more insight into the $\Omega^{(\mathrm{YM})}$-term on the second line of equation (5.24), it is useful to expand the superfield expression on the right hand side into its field components.[6] We obtain

$$\begin{aligned}\delta(\Lambda)\mathcal{L}_{\mathrm{g.k.}} &= \frac{1}{2}X_{111}\left[\mathrm{i}e\left(\partial_\mu\Lambda\right)\bar{\lambda}\gamma^5\gamma_\mu\lambda+\frac{1}{4}\Lambda\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right] \\ &= \delta(\Lambda)\left(\frac{1}{4}\mathrm{i}eX_{111}A_\mu\bar{\lambda}\gamma^5\gamma^\mu\lambda\right)+\frac{1}{8}\Lambda X_{111}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\,. \end{aligned} \tag{5.25}$$

The first term is given by the gauge variation of a bilinear expression in the gaugino. This contribution is an artefact of the superfield formalism and the Wess-Zumino gauge. In fact, the bilinear expression can be absorbed in the kinetic term $\frac{1}{4}\mathrm{i}e(\partial_\mu \operatorname{Im} f_{11})\bar{\lambda}\gamma^5\gamma^\mu\lambda$ in $\mathcal{L}_{\mathrm{g.k.}}$ (recall equation (3.47)), such that the ordinary derivative on the gauge kinetic function becomes a gauge covariant derivative:

$$\frac{1}{4}\mathrm{i}e(\partial_\mu \operatorname{Im} f_{11})\bar{\lambda}\gamma^5\gamma^\mu\lambda-\frac{1}{4}\mathrm{i}eX_{111}A_\mu\bar{\lambda}\gamma^5\gamma^\mu\lambda=\frac{1}{4}\mathrm{i}e(D_\mu \operatorname{Im} f_{11})\bar{\lambda}\gamma^5\gamma^\mu\lambda\,. \tag{5.26}$$

If we make this change to $\mathcal{L}_{\mathrm{g.k.}}$ and denote the new Lagrangian with $\hat{\mathcal{L}}_{\mathrm{g.k.}}$, the variation of the corresponding action $\hat{S}_{\mathrm{g.k.}}=\int \mathrm{d}^4x\,\hat{\mathcal{L}}_{\mathrm{g.k.}}$ reduces to

$$\delta(\Lambda)\hat{S}_{\mathrm{g.k.}}=\frac{1}{8}\int \mathrm{d}^4x\,\Lambda X_{111}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\,. \tag{5.27}$$

The variation is quadratic in the Abelian field strengths and has exactly the same form as the gauge anomaly in (2.73):

$$\mathcal{A}_{\mathrm{gauge}}=-\frac{1}{8}\int \mathrm{d}^4x\,\Lambda d_{111}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\,, \tag{5.28}$$

where $d_{111}$ is defined in (2.74) to be a cubic expression in the U(1) charges of the right-handed fermions:[7] $d_{111}=(1/12\pi^2)\sum_i q_i^3$. If the condition $X_{111}=$

[6] We impose the Wess-Zumino gauge fixing condition. The meaning and consequences of this non-trivial ansatz will be discussed at length in §5.3.

[7] In (2.74) we sum over all chiral fermions, but do not include particles and anti-particles separately. Since any left-handed particle can equivalently be described by its right handed anti-particle, we can restrict the sum to right handed (anti-)fermions only.

$d_{111}$ is satisfied, we see that the classical variation $\delta(\Lambda)\hat{S}_{\text{g.k.}}$ cancels the quantum anomaly. This is the easiest example of the $4D$ Green-Schwarz mechanism. Due to the particular form of $d_{111}$, this anomaly cancellation mechanism puts strong constraints on the mutual combination of $m$ and the charges $q_i$:

$$12\pi^2 m = \sum_i q_i^3\,. \tag{5.29}$$

**2.** For the $\Omega^{(\mathrm{L})}$-term in (5.24), it is useful to return to the original form in terms of the Weyl multiplet:

$$\delta(\Lambda)\mathcal{L}_{\text{Weyl}} = \int \mathrm{d}^2\theta\,\mathcal{E}\,\left(-\Lambda X_{111} W_{\mu\nu\alpha} W_{\beta}^{\mu\nu}\varepsilon^{\alpha\beta}\right) + \text{h.c.}. \tag{5.30}$$

Again we can write down the component expression:

$$\delta(\Lambda)\mathcal{L}_{\text{Weyl}} = -\frac{1}{4}\Lambda X_{111}\varepsilon^{\mu\nu\rho\sigma} R_{\mu\nu\,ab} R_{\rho\sigma}{}^{ab} + \dots\,. \tag{5.31}$$

The dots denote extra terms that will be omitted for the moment, but will be considered in §5.7. The first term has exactly the form of the mixed gauge-gravitational anomaly as it follows from the gauge variation of the effective action [24]:

$$\mathcal{A}_{\text{mixed}} = \frac{1}{192\pi^2}\int \mathrm{d}^4x\,\Lambda\left(\sum_i q_i\right)\varepsilon^{\mu\nu\rho\sigma} R_{\mu\nu\,ab} R_{\rho\sigma}{}^{ab}\,. \tag{5.32}$$

This anomaly can therefore be canceled if the condition

$$-48\pi^2 m = \sum_i q_i \tag{5.33}$$

is satisfied. Again this puts severe constraints on the combination of charges and the constant $m$.

We conclude that for the easiest 4-dimensional examples, the Peccei-Quinn term and Hirzebruck signature density play the role of counterterms in the Green-Schwarz mechanism, and a cancellation of the anomaly is only possible if the axion field transforms with a shift symmetry. However, in light of the 10-dimensional Green-Schwarz mechanism in §5.2, a more natural formulation would be to study the "dual" mechanism in terms of an antisymmetric 2-form and its corresponding gauge transformation. This will be the content of the last short section before we go on to study the 4-dimensional Green-Schwarz mechanism in full detail.

### Dual Lagrangian

The dual formulation of the Green-Schwarz mechanism can be obtained upon dualization of the chiral multiplet $S$ to a linear multiplet $L$ (i.e., a real multiplet satisfying the constraints $\mathcal{D}^2 L = \bar{\mathcal{D}}^2 L = 0$). The details of this construction are as follows. One starts from a first order form, connecting the two dual theories:[8]

$$\mathcal{L}_{\text{1st order}} = \int \mathrm{d}^4\theta \Big[ K[U - mV] - LU + U \left( \Omega^{(\text{YM})} - \Omega^{(\text{L})} \right) \Big] , \tag{5.34}$$

where $K$ is the Kähler potential that depends on the combination $U - mV$, and $U$ is a real superfield. A variation of $\mathcal{L}_{\text{1st order}}$ with respect to $L$ gives $U = S + \bar{S}$ and after a substitution we recover the original Lagrangian (5.20) (we did not include the kinetic term in (5.20)):

$$\mathcal{L}_{\text{4D}} = \int \mathrm{d}^4\theta \Big[ K[S + \bar{S} - mV] + \left( S + \bar{S} \right) \left( \Omega^{(\text{YM})} - \Omega^{(\text{L})} \right) \Big] . \tag{5.35}$$

The dual Lagrangian instead is obtained by varying $\mathcal{L}_{\text{1st order}}$ with respect to $U$. Let us introduce the new notations

$$\tilde{U} = U - mV , \qquad \tilde{L} = L - \Omega^{(\text{YM})} + \Omega^{(\text{L})} . \tag{5.36}$$

Then the relevant terms are

$$\mathcal{L}_{\text{1st order}} = \int \mathrm{d}^4\theta \Big[ K[\tilde{U}] - \tilde{L}\tilde{U} - \tilde{L} mV \Big] . \tag{5.37}$$

Solving

$$\Psi \equiv \frac{\partial K[\tilde{U}]}{\partial \tilde{U}} - \tilde{L} = 0 , \tag{5.38}$$

one gets

$$\mathcal{L}_{\text{dual}} = \int \mathrm{d}^4\theta \Big[ \Phi[\tilde{L}] - \tilde{L} mV \Big] , \tag{5.39}$$

where $\Phi[\tilde{L}] = K[\tilde{U}] - \tilde{L}\tilde{U}$ at $\Psi = 0$. The first term in $\mathcal{L}_{\text{dual}}$ contains the kinetic Lagrangian for the fields in the linear multiplet. In particular, there is a $\tilde{H}_{\mu\nu\rho}\tilde{H}^{\mu\nu\rho}$-term with

$$\tilde{H}_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]} - \Omega^{(\text{YM})}_{\mu\nu\rho} + \Omega^{(\text{L})}_{\mu\nu\rho} \tag{5.40}$$

the (modified) field strength of the antisymmetric 2-form. This field strength is contained in the $\theta\bar{\theta}$-component of $\tilde{L}$, and it resembles the 10-dimensional expression (5.9).

The second part of (5.39) is the Green-Schwarz counterterm; its field expansion contains a contribution $mB_{\mu\nu}F_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}$ which is the 4-dimensional equivalent of (5.6).

[8] We closely follow the procedure outlined in [12], but extend their treatment to include the Lorentz Chern-Simons form.

Finally, the gauge and local Lorentz transformations of $L$ can be obtained from the requirement that $\tilde{L}$ (and therefore $\tilde{H}_{\mu\nu\rho}$) is invariant. For example, from (5.15) and (5.36) it follows that

$$\delta(\Lambda)L = \delta(\Lambda)\Omega^{(\mathrm{YM})} = \mathcal{D}^{\alpha}\left(\Lambda W_{\alpha}\right) + \bar{\mathcal{D}}_{\dot{\alpha}}\left(\bar{\Lambda}\bar{W}^{\dot{\alpha}}\right) .$$

In particular, the component 2-form $B_{\mu\nu}$ transforms as

$$\delta(\Lambda,\lambda)B_{\mu\nu} = 2\Lambda\partial_{[\mu}A_{\nu]} - 2\lambda_{ab}\partial_{[\mu}\omega_{\nu]}{}^{ab} . \tag{5.41}$$

This variation is identical to (5.7). Of course, these transformations do not leave the total Lagrangian $\mathcal{L}_{\mathrm{dual}}$ invariant, but they lead to the following non-vanishing result:

$$\begin{aligned} \delta(\Lambda,\lambda)\mathcal{L}_{\mathrm{dual}} &= \int \mathrm{d}^4\theta \Big[ -\tilde{L}m\left(\Lambda+\bar{\Lambda}\right)\Big] \\ &= \int \mathrm{d}^4\theta\, m\left(\Lambda+\bar{\Lambda}\right)\left(\Omega^{(\mathrm{YM})}-\Omega^{(\mathrm{L})}\right), \end{aligned} \tag{5.42}$$

which is identical to (5.24), as expected. Under the same conditions (5.29) and (5.33), this classical non-invariance cancels the quantum anomaly.

Let us wrap up this section with a brief summary of our results so far. We have studied two alternative formulations of the 4-dimensional GS anomaly cancellation mechanism, inspired by an elementary string theory reduction in §5.2. In terms of the chiral multiplet, the counterterms are a product of the axion field and a topological quantity, namely $F \wedge F$ and the Hirzebruck signature density. Under a gauged shift transformation of the axion, $\operatorname{Im} z \to \operatorname{Im} z + m\Lambda(x)$, these counterterms take the form of the gauge and mixed anomaly respectively. However, the cancellation only occurs if the parameter $m$ and the charges in the theory satisfy the consistency conditions (5.29) and (5.33). A similar mechanism hold for the dual formulation, which is more natural from a (higher-dimensional) string theory perspective. Here, the counterterms are proportional to the $B_{\mu\nu}$ and the latter transforms as $B \to B + \Lambda dA - \lambda d\omega$.

In the remainder of this chapter, these results will be generalized in several directions. We analyze the variation of a generic PQ term under shift transformations of the gauge kinetic function $f_{\Lambda\Sigma}$. This leads to a non-invariance that can be canceled by *two* mechanisms, or a combination thereof: (i) generalized Chern-Simons terms and (ii) gauge anomalies. The former will be introduced in §5.3, the cancellation of the anomalies is discussed at length in §5.4. After that, the analysis is repeated for $\mathcal{N}=1$ supergravity. The results about gauge anomalies can be extended straightforwardly, and will be supplied by an analysis of mixed gauge-gravitational anomaly cancellation in §5.7.

## 5.3 Kinetic and Chern-Simons action

### Kinetic action and the Wess - Zumino gauge

The vector multiplet in the $\mathcal{N} = 1$ superspace formulation is described by a real superfield. The latter has many more components than the physical fields describing an on-shell vector multiplet, which consists of one vector field and one fermion (recall Table 3.1). The advantage of this redundancy is that one can easily construct manifestly supersymmetric actions as integrals over full or chiral superspace. We saw already an example in (5.12), which has the following generalization:[9]

$$S_f = \int \mathrm{d}^4x\mathrm{d}^2\theta\, f_{\Lambda\Sigma}(S) W_\alpha^\Lambda W_\beta^\Sigma \varepsilon^{\alpha\beta} + \text{h.c.} \tag{5.43}$$

Here, $W_\alpha^\Lambda = \frac{1}{4}\bar{\mathcal{D}}^2\mathcal{D}_\alpha V^\Lambda$, or a generalization thereof for the non-Abelian case, where $V^\Lambda$ is the real superfield describing the vector multiplets labeled by an index $\Lambda$. As usual, $f_{\Lambda\Sigma}$ is an arbitrary holomorphic function of a set of chiral superfields denoted by $S$.

The integrand of (5.43) is itself a chiral superfield. As we integrate over a chiral superspace, the Lagrangian transforms into a total derivative under local supersymmetry variations and $S_f$ is invariant. Formally, this conclusion holds independently of the gauge symmetry properties of the functions $f_{\Lambda\Sigma}$. For the action (5.43) to be gauge invariant, though, we should have the transformations [8]

$$\begin{aligned} &V^\Sigma \to V^\Sigma + \Lambda^\Sigma + \bar{\Lambda}^\Sigma\,, \\ &\delta(\Lambda) f_{\Lambda\Sigma} - 2\Lambda^\Omega(x) f_{\Omega(\Lambda}{}^\Omega f_{\Sigma)\Omega} = 0\,, \end{aligned} \tag{5.44}$$

where $\Lambda^\Sigma$ is a chiral superfield and its $\bar\theta\theta$-component is given by the derivative of the local gauge parameter $\Lambda^\Sigma(x)$. Remark that the transformation of $f_{\Lambda\Sigma}$ in (5.44) is less general than (4.48) which followed from symplectic arguments. In particular, the constant shift tensor $X_{\Omega\Lambda\Sigma}$ is absent.

Due to the large number of fields in the superspace formulation, the gauge parameters $\Lambda^\Sigma$ are not just real numbers, but are themselves full chiral superfields. To describe the physical theory, one wants to get rid of these extra gauge transformations and thereby also of many spurious components of the vector superfields. This is done by going to the so-called Wess-Zumino gauge [110], in which these extra gauge transformations are fixed and many spurious components of the real superfields are eliminated. Unfortunately, the Wess-Zumino gauge also breaks the manifest supersymmetry of the superspace formalism. However, a combination of this original "superspace supersymmetry" and the gauge

[9] Since we restrict to global supersymmetry, we have not included the chiral density $\mathcal{E}$ that is necessary in order to obtain the correct expression in curved space-time.

symmetries survives and becomes the preserved supersymmetry after the gauge fixing. The law that gives the preserved supersymmetry as a combination of these different symmetries is called the "decomposition law", see e.g. eq. (2.28) in [8]. Notice, however, that this preservation requires the gauge invariance of the original action (5.43). Thus, though (5.43) was invariant under the superspace supersymmetry for any choice of $f_{\Lambda\Sigma}$, we now need (5.44) for this action to be invariant under supersymmetry after the Wess-Zumino gauge.

This important consequence of the Wess-Zumino gauge can also be understood from the supersymmetry algebra. The superspace operator $Q_\alpha$ satisfies the anticommutation relation

$$\left\{Q_\alpha, Q^\dagger_{\dot\alpha}\right\} = \sigma^\mu_{\alpha\dot\alpha}\partial_\mu\,. \tag{5.45}$$

This equation shows no mixing between supersymmetry and gauge symmetries. However, after the Wess-Zumino gauge the right-hand side is changed to [111]

$$\left\{Q_\alpha, Q^\dagger_{\dot\alpha}\right\} = \sigma^\mu_{\alpha\dot\alpha}D_\mu = \sigma^\mu_{\alpha\dot\alpha}\left(\partial_\mu - A_\mu{}^\Lambda\delta_\Lambda\right)\,, \tag{5.46}$$

where $\delta_\Lambda$ denotes the gauge transformation. Equation (5.46) implies that if an action is invariant under supersymmetry, it should also be gauge invariant.

As mentioned before, the preservation of the Wess-Zumino gauges implies that the effective supersymmetry transformations are different from the ones in the original superspace formulation, see (3.34). It is shown in [111] that the resulting supersymmetry transformations of a chiral multiplet are

$$\begin{aligned}
\delta(\epsilon)z^i &= \bar\epsilon_{(L)}\chi^i_{(L)}\,,\\
\delta(\epsilon)\chi^i_{(L)} &= \frac{1}{2}\gamma^\mu\epsilon_{(R)}D_\mu z^i + \frac{1}{2}h^i\epsilon_{(L)}\,,\\
\delta(\epsilon)h^i &= \bar\epsilon_{(R)}\gamma^\mu D_\mu\chi^i_{(L)} + \bar\epsilon_{(R)}\lambda^\Sigma_{(R)}\delta_\Sigma z^i\,.
\end{aligned} \tag{5.47}$$

These transformations are valid for *any* chiral multiplet, in particular, they can be applied to the full integrand of (5.43) itself. We will make use of this in §5.3.

Compared to the standard superspace transformations in (3.34), there are two modifications in (5.47). The first modification is that the derivatives of $z^i$ and $\chi^i_{(L)}$ are covariantized with respect to gauge transformations. The action of the covariant derivative on the scalars was given in (4.54), the action on the chiral fermions $\chi^i_{(L)}$ can be determined from (4.53) and (4.13):

$$D_\mu\chi^i_{(L)} = \left(\partial_\mu - A_\mu{}^\Lambda\delta_\Lambda\right)\chi^i_{(L)} = \left(\partial_\mu\delta^i_j - A_\mu{}^\Lambda\partial_j k^i_\Lambda\right)\chi^j_{(L)}\,. \tag{5.48}$$

The second modification is the additional last term in the transformation of the auxiliary fields $h^i$. The origin of this term lies in the contribution of the decomposition law for one of the gauge symmetries contained in the chiral superfield of transformations $\Lambda^{\Sigma}$, after the Wess-Zumino gauge is fixed.

To avoid the above-mentioned subtleties associated with the Wess-Zumino gauge, we will use component field expressions in the remainder of this text. Therefore, we reconsider the action (5.43) and in particular its integrand. The components of this composite chiral multiplet are [8]

$$
\begin{aligned}
z(fW^2) &= -\frac{1}{2} f_{\Lambda\Sigma} \bar{\lambda}^{\Lambda}_{(L)} \lambda^{\Sigma}_{(L)} \,, \\
\chi_{(L)}(fW^2) &= \frac{1}{2} f_{\Lambda\Sigma} \left( \frac{1}{2} \gamma^{\mu\nu} \mathcal{F}_{\mu\nu}{}^{\Lambda} - \mathrm{i} D^{\Lambda} \right) \lambda^{\Sigma}_{(L)} - \frac{1}{2} \partial_i f_{\Lambda\Sigma} \chi^i_{(L)} \bar{\lambda}^{\Lambda}_{(L)} \lambda^{\Sigma}_{(L)} \,, \\
h(fW^2) &= f_{\Lambda\Sigma} \left( -\bar{\lambda}^{\Lambda}_{(L)} \gamma^{\mu} D_{\mu} \lambda^{\Sigma}_{(R)} - \frac{1}{2} \mathcal{F}^{-\Lambda}_{\mu\nu} \mathcal{F}^{\mu\nu\, -\Sigma} + \frac{1}{2} D^{\Lambda} D^{\Sigma} \right) \\
&\quad + \partial_i f_{\Lambda\Sigma} \chi^i_{(L)} \left( -\frac{1}{2} \gamma^{\mu\nu} \mathcal{F}_{\mu\nu}{}^{\Lambda} + \mathrm{i} D^{\Lambda} \right) \lambda^{\Sigma}_{(L)} \\
&\quad - \frac{1}{2} \partial_i f_{\Lambda\Sigma} h^i \bar{\lambda}^{\Lambda}_{(L)} \lambda^{\Sigma}_{(L)} + \frac{1}{2} \partial^2_{ij} f_{\Lambda\Sigma} \bar{\chi}^i_{(L)} \chi^j_{(L)} \bar{\lambda}^{\Lambda}_{(L)} \lambda^{\Sigma}_{(L)} \,.
\end{aligned}
\tag{5.49}
$$

The superspace integral in (5.43) means that the real part of $h(fW^2)$ is (proportional to) the Lagrangian:

$$
S_f = \int \mathrm{d}^4 x \; \mathrm{Re}\, h(fW^2) \,. \tag{5.50}
$$

From (5.49) and (5.50), we read off the kinetic terms of $S_f$:

$$
\begin{aligned}
S_{f,\,\mathrm{kin}} &= \int \mathrm{d}^4 x \Big[ -\frac{1}{4} \mathrm{Re}\, f_{\Lambda\Sigma} \mathcal{F}_{\mu\nu}{}^{\Lambda} \mathcal{F}^{\mu\nu\Sigma} - \frac{1}{2} \mathrm{Re}\, f_{\Lambda\Sigma} \bar{\lambda}^{\Lambda} \gamma^{\mu} D_{\mu} \lambda^{\Sigma} \qquad (5.51) \\
&\qquad + \frac{1}{4} \mathrm{i}\, \mathrm{Im}\, f_{\Lambda\Sigma} \mathcal{F}_{\mu\nu}{}^{\Lambda} \tilde{\mathcal{F}}^{\mu\nu\Sigma} + \frac{1}{4} \mathrm{i} (D_{\mu} \mathrm{Im}\, f_{\Lambda\Sigma}) \bar{\lambda}^{\Lambda} \gamma^5 \gamma^{\mu} \lambda^{\Sigma} \Big] \,,
\end{aligned}
$$

which is the same action as in (3.39) but with additional minimal couplings to the vectors. This manifests itself via the covariant field strengths $\mathcal{F}_{\mu\nu}{}^{\Lambda}$ instead of $F_{\mu\nu}{}^{\Lambda}$, and covariant derivatives $D_{\mu}$ instead of $\partial_{\mu}$. In particular, the covariant derivative acting on ($\mathrm{Im}\, f_{\Lambda\Sigma}$) is defined via

$$
D_{\mu} f_{\Lambda\Sigma} = \partial_{\mu} f_{\Lambda\Sigma} - 2 A_{\mu}{}^{\Omega} f_{\Omega(\Lambda}{}^{\Xi} f_{\Sigma)\Xi} \,. \tag{5.52}
$$

In the case that the gauge kinetic matrix transforms without a shift, as in (5.44), the derivative defined in (5.52) is fully gauge covariant.

In §4.1, we motivated a more general gauge transformation rule for $f_{\Lambda\Sigma}$, in which shifts proportional to $X_{\Omega\Lambda\Sigma}$ are allowed[10] as in (4.48). Then (5.52) is no longer the full covariant derivative. The full covariant derivative has instead the new form

$$\hat{D}_\mu f_{\Lambda\Sigma} \equiv \partial_\mu f_{\Lambda\Sigma} - A_\mu{}^\Omega \delta_\Omega f_{\Lambda\Sigma} = D_\mu f_{\Lambda\Sigma} - \mathrm{i} A_\mu{}^\Omega X_{\Omega\Lambda\Sigma} \,. \tag{5.53}$$

The last term in (5.51) is therefore not gauge covariant for non-vanishing $X_{\Omega\Lambda\Sigma}$. Hence, in presence of a shift transformation of $f_{\Lambda\Sigma}$, we replace the action $S_f$ with $\hat{S}_f$, in which we use the full covariant derivative, $\hat{D}_\mu$, instead of $D_\mu$. More precisely, we define

$$\hat{S}_f = S_f + S_{\text{extra}} \,, \qquad S_{\text{extra}} = \int \mathrm{d}^4 x \left( -\frac{1}{4} \mathrm{i} A_\mu{}^\Omega X_{\Omega\Lambda\Sigma} \bar{\lambda}^\Lambda \gamma_5 \gamma^\mu \lambda^\Sigma \right) \,. \tag{5.54}$$

Note that we did not use any superspace expression to derive $S_{\text{extra}}$ but simply added $S_{\text{extra}}$ by hand in order to fully covariantize the last term of (5.51). As we will further discuss in the next section, $S_{\text{extra}}$ can in fact only be partially understood from superspace expressions, which motivates our procedure to introduce it here by hand. We should also stress that the covariantization with $S_{\text{extra}}$ does not yet mean that the entire action $\hat{S}_f$ is now fully gauge invariant. The gauge and supersymmetry transformations of $\hat{S}_f$ will be discussed in the next section.

### Gauge and supersymmetry transformations

The action $S_f$ is gauge invariant before the modification of the transformation of $f_{\Lambda\Sigma}$. In the presence of the $X_{\Omega\Lambda\Sigma}$ terms, the action $\hat{S}_f$ is not gauge invariant. However, the non-invariance comes only from one term. Indeed, terms in $\hat{S}_f$ that are proportional to derivatives of $f_{\Lambda\Sigma}$ do not feel the constant shift $\delta_\Omega f_{\Lambda\Sigma} = \mathrm{i} X_{\Omega\Lambda\Sigma} + \ldots$. They are therefore automatically gauge invariant. Also, the full covariant derivative (5.53) has no gauge transformation proportional to $X_{\Omega\Lambda\Sigma}$, and also $\operatorname{Re} f_{\Lambda\Sigma}$ is invariant. Hence, the gauge non-invariance originates only from the third term in (5.51). We are thus left with

$$\delta(\Lambda) \hat{S}_f = \frac{1}{8} X_{\Omega\Lambda\Sigma} \int \mathrm{d}^4 x \, \Lambda^\Omega \varepsilon^{\mu\nu\rho\sigma} \mathcal{F}_{\mu\nu}{}^\Lambda \mathcal{F}_{\rho\sigma}{}^\Sigma \,. \tag{5.55}$$

This expression is equal to the result we found in (4.58), but we have now proven it is the only non-vanishing contribution to the gauge variation of the action.

We started to construct $S_f$ as a superspace integral, and as such it would automatically be supersymmetric. However, we saw that when $f_{\Lambda\Sigma}$ transforms with a shift, the gauge symmetry is broken, which is then communicated to the

[10] Recall that $X_{\Omega\Lambda\Sigma}$ is symmetric in its last two lower indices: $X_{\Omega\Lambda\Sigma} = X_{\Omega\Sigma\Lambda}$

supersymmetry transformations by the Wess-Zumino gauge fixing. The $X_{\Omega\Lambda\Sigma}$ tensors then express the non-invariance of $S_f$ under both gauge transformations and supersymmetry.

To determine these supersymmetry transformations, we consider the last line of (5.47) for $\{z^i, \chi^i, h^i\}$ replaced by $\{z(fW^2), \chi(fW^2), h(fW^2)\}$ and find

$$\begin{aligned}\delta(\epsilon)S_f &= \int \mathrm{d}^4x\,\mathrm{Re}\left[\bar\epsilon_{(R)}\gamma^\mu\partial_\mu\chi_{(L)}(fW^2) - \bar\epsilon_{(R)}\gamma^\mu A_\mu{}^\Sigma\delta_\Sigma\chi_{(L)}(fW^2)\right.\\ &\qquad\left. + \bar\epsilon_{(R)}\lambda^\Sigma_{(R)}\delta_\Sigma z(fW^2)\right]. \end{aligned} \tag{5.56}$$

The first term in the transformation of $h(fW^2)$ is the one that was already present in the superspace supersymmetry before going to Wess-Zumino gauge. It is a total derivative, as we would expect from the superspace rules. The other two terms are due to the mixing of supersymmetry with gauge symmetries. They vanish if $z(fW^2)$ is invariant under the gauge symmetry, as this implies by (4.13) that $\chi(fW^2)$ is also gauge invariant.

Using (4.48) and (5.49), however, one sees that $z(fW^2)$ is not gauge invariant, and (5.56) becomes, using also (4.13),

$$\begin{aligned}\delta(\epsilon)S_f &= \int \mathrm{d}^4x\,\mathrm{Re}\left[-\mathrm{i}X_{\Omega\Lambda\Sigma}\bar\epsilon_{(R)}\gamma^\mu A_\mu{}^\Omega\left(\frac{1}{4}\gamma^{\rho\sigma}\mathcal{F}_{\rho\sigma}{}^\Lambda - \frac{1}{2}\mathrm{i}D^\Lambda\right)\lambda^\Sigma_{(L)}\right.\\ &\qquad\left. - \frac{1}{2}\mathrm{i}X_{\Omega\Lambda\Sigma}\bar\epsilon_{(R)}\lambda^\Omega_{(R)}\bar\lambda^\Lambda_{(L)}\lambda^\Sigma_{(L)}\right]. \end{aligned} \tag{5.57}$$

Note that this expression contains only fields of the vector multiplets and none of the chiral multiplets.

It remains to determine the contribution of $S_{\text{extra}}$ to the supersymmetry variation, which turns out to be

$$\begin{aligned}\delta(\epsilon)S_{\text{extra}} &= \int \mathrm{d}^4x\,\mathrm{Re}\left[-\frac{1}{2}\mathrm{i}X_{\Omega\Lambda\Sigma}A_\mu{}^\Omega\bar\lambda^\Sigma_{(L)}\gamma^\mu\left(\frac{1}{2}\gamma^{\nu\rho}\mathcal{F}_{\nu\rho}{}^\Lambda - \mathrm{i}D^\Lambda\right)\epsilon_{(R)}\right.\\ &\qquad\left. - \mathrm{i}X_{\Omega\Lambda\Sigma}\bar\epsilon_{(R)}\lambda^\Sigma_{(R)}\bar\lambda^\Omega_{(L)}\lambda^\Lambda_{(L)}\right]. \end{aligned} \tag{5.58}$$

By combining this with (5.57), we obtain, after some reordering,

$$\begin{aligned}\delta(\epsilon)\hat S_f &= \int \mathrm{d}^4x\,\mathrm{Re}\left(\frac{1}{2}X_{\Omega\Lambda\Sigma}\varepsilon^{\mu\nu\rho\sigma}A_\mu{}^\Omega\mathcal{F}_{\nu\rho}{}^\Lambda\bar\epsilon_{(R)}\gamma_\sigma\lambda^\Sigma_{(L)}\right.\\ &\qquad\left. - \frac{3}{2}\mathrm{i}X_{(\Omega\Lambda\Sigma)}\bar\epsilon_{(R)}\lambda^\Omega_{(R)}\bar\lambda^\Lambda_{(L)}\lambda^\Sigma_{(L)}\right). \end{aligned} \tag{5.59}$$

In order to understand how this broken gauge and supersymmetry invariance can be restored, it is convenient to split the coefficients $X_{\Omega\Lambda\Sigma}$ into a sum,

$$X_{\Omega\Lambda\Sigma} = X^{(\mathrm{s})}_{\Omega\Lambda\Sigma} + X^{(\mathrm{m})}_{\Omega\Lambda\Sigma}\,, \qquad X^{(\mathrm{s})}_{\Omega\Lambda\Sigma} = X_{(\Omega\Lambda\Sigma)}\,, \qquad X^{(\mathrm{m})}_{(\Omega\Lambda\Sigma)} = 0\,, \tag{5.60}$$

where $X^{(\mathrm{s})}_{\Omega\Lambda\Sigma}$ is completely symmetric and $X^{(\mathrm{m})}_{\Omega\Lambda\Sigma}$ denotes the part of mixed symmetry.[11] The non-vanishing transformations in (5.55) and (5.59) may then in principle be canceled by the following two mechanisms, or a combination thereof:

(i) As was first realized in a similar context in $\mathcal{N}=2$ supergravity in [41] (see also the systematic analysis [112]) and later for rigid $\mathcal{N}=1$ theories in [12], the gauge variation due to a non-vanishing mixed part, $X^{(\mathrm{m})}_{\Omega\Lambda\Sigma} \neq 0$, may be canceled by adding a generalized Chern-Simons term (GCS term) that contains a cubic and a quartic part in the vector fields:

$$S_{\mathrm{CS}} = \frac{1}{2} X^{(\mathrm{CS})}_{\Omega\Lambda\Sigma} \int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma} \left( \frac{1}{3} A_\mu{}^\Omega A_\nu{}^\Lambda F_{\rho\sigma}{}^\Sigma + \frac{1}{4} f_{\Xi\Upsilon}{}^\Lambda A_\mu{}^\Xi A_\nu{}^\Upsilon A_\rho{}^\Omega A_\sigma{}^\Sigma \right). \tag{5.61}$$

This action depends on a constant tensor $X^{(\mathrm{CS})}_{\Omega\Lambda\Sigma}$, which has also a mixed symmetry structure, i.e.,

$$X^{(\mathrm{CS})}_{(\Omega\Lambda\Sigma)} = 0\,. \tag{5.62}$$

The cancellation occurs provided the tensors $X^{(\mathrm{m})}_{\Omega\Lambda\Sigma}$ and $X^{(\mathrm{CS})}_{\Omega\Lambda\Sigma}$ are the same. The details of this result will be reviewed in §5.3.

(ii) If the chiral fermion spectrum is anomalous under the gauge group, the anomalous triangle diagrams lead to a non-gauge invariance of the quantum effective action of the form (2.73) with a symmetric tensor $d_{\Omega\Lambda\Sigma}$. If $X^{(\mathrm{s})}_{\Omega\Lambda\Sigma} = d_{\Omega\Lambda\Sigma}$, the quantum anomaly cancels the symmetric part of (5.55).[12] This is the Green-Schwarz mechanism.

In combination with the gauge anomaly, there is also a supersymmetry anomaly, because $\delta(\Lambda)\Gamma \neq 0$ and (5.46) imply that $\delta(\epsilon)\Gamma \neq 0$. The supersymmetry anomaly is proportional to the same symmetric coefficients $d_{\Omega\Lambda\Sigma}$, and a cancellation of the non-zero variation in (5.59) occurs if the equality $X^{(\mathrm{s})}_{\Omega\Lambda\Sigma} = d_{\Omega\Lambda\Sigma}$ holds. The details of the GS mechanism will be discussed in §5.4.

[11] This corresponds to the decomposition $\square\square \otimes \square = \square\square\square \oplus \begin{smallmatrix}\square\square\\ \square\phantom{\square}\end{smallmatrix}$.

[12] It should be noted that this situation is qualitatively different from the analogous treatment in the context of extended supersymmetry. There one can show that $X^{(\mathrm{s})}_{\Omega\Lambda\Sigma}$ vanishes identically (recall footnote 11 in §4.2), but there are also no chiral gauge interactions and hence no quantum anomalies.

### Chern-Simons action

Due to the gauged shift symmetry of $f_{\Lambda\Sigma}$, terms proportional to $X_{\Omega\Lambda\Sigma}$ remain in the gauge and supersymmetry variation of the action $\hat{S}_f$. To re-establish the gauge symmetry and supersymmetry invariance, we need two ingredients: GCS terms and quantum anomalies.

The former were in part already discussed in equation (5.61). As was described in [12], they can be obtained from a superfield expression:

$$\begin{aligned} S'_{\rm CS} \;=\; & X^{(\rm CS)}_{\Omega\Lambda\Sigma}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\Big[-\frac{2}{3}V^{\Omega}\Omega^{\Lambda\Sigma}_{(\rm YM)}(V) \\ & + \left(f_{\Xi\Upsilon}{}^{\Sigma}V^{\Omega}\mathcal{D}^{\alpha}V^{\Lambda}\bar{\mathcal{D}}^2\left(\mathcal{D}_{\alpha}V^{\Xi}V^{\Upsilon}\right)+\text{h.c.}\right)\Big]. \end{aligned} \qquad (5.63)$$

and an expression for $\Omega^{\Lambda\Sigma}_{(\rm YM)}$ in terms of the real superfields was given already in (5.5).

The full non-Abelian superspace expression (5.63) is valid only in the Wess-Zumino gauge, where it reduces to the bosonic component expression (5.61) plus a fermionic term [12]:

$$S'_{\rm CS}=S_{\rm CS}+(S'_{\rm CS})_{\rm ferm}\,,\quad (S'_{\rm CS})_{\rm ferm}=\int \mathrm{d}^4x\left(-\frac{1}{4}\mathrm{i}X^{(\rm CS)}_{\Omega\Lambda\Sigma}A_{\mu}{}^{\Omega}\bar{\lambda}^{\Lambda}\gamma_5\gamma^{\mu}\lambda^{\Sigma}\right), \qquad (5.64)$$

where we used the restriction $X^{(\rm CS)}_{(\Omega\Lambda\Sigma)}=0$ from (5.62).

Note that the fermionic term in (5.64) is of a form similar to $S_{\rm extra}$ in (5.54). More precisely, in (5.64) the fermions appear with the tensor $X^{(\rm CS)}_{\Omega\Lambda\Sigma}$, which has a mixed symmetry. On the other hand, $S_{\rm extra}$ in (5.54) is proportional to the tensor $X^{(s)}_{\Omega\Lambda\Sigma}+X^{(m)}_{\Omega\Lambda\Sigma}$. From this we see that if we identify $X^{(m)}_{\Omega\Lambda\Sigma}=X^{(\rm CS)}_{\Omega\Lambda\Sigma}$, as we will do later, we can absorb the mixed part of $S_{\rm extra}$ into the superspace expression $S'_{\rm CS}$. This is, however, not possible for the symmetric part of $S_{\rm extra}$ proportional to $X^{(s)}_{\Omega\Lambda\Sigma}$, which cannot be obtained in any obvious way from a superspace expression. As we need this symmetric part later, it is more convenient to keep the full $S_{\rm extra}$, as we did in §5.3, as a part of $\hat{S}_f$, and not include $(S'_{\rm CS})_{\rm ferm}$ here. Thus, we will further work with the purely bosonic $S_{\rm CS}$ and omit the fermionic term that is included in the superspace expression (5.63).

The GCS term $S_{\rm CS}$ is not gauge invariant. Even the superspace expression $S'_{\rm CS}$ is not gauge invariant, not even in the Abelian case. So, just as for $S_f$, we expect that $S'_{\rm CS}$ is not supersymmetric in the Wess-Zumino gauge, despite the fact that it is a superspace integral. This is highlighted, in particular, by the second term in (5.63), which involves the structure constants. Its component expression simply gives the non-Abelian $A\wedge A\wedge A\wedge A$ correction in (5.61), which, as a purely bosonic object, cannot be supersymmetric by itself.

For the gauge variation of $S_{\rm CS}$, one obtains

$$\begin{aligned}
\delta(\Lambda)S_{\rm CS} = & \\
\int \mathrm{d}^4x \Big[ & -\frac{1}{4}\mathrm{i}X^{(\rm CS)}_{\Omega\Lambda\Sigma}\Lambda^\Omega F_{\mu\nu}{}^\Lambda \tilde F^{\mu\nu\Sigma} \qquad (5.65)\\
& -\frac{1}{8}\Lambda^\Omega\Big(2X^{(\rm CS)}_{\Xi\Lambda\Sigma}f_{\Omega\Upsilon}{}^\Sigma - X^{(\rm CS)}_{\Sigma\Xi\Lambda}f_{\Omega\Upsilon}{}^\Sigma + X^{(\rm CS)}_{\Xi\Sigma\Upsilon}f_{\Omega\Lambda}{}^\Sigma\\
&\qquad - X^{(\rm CS)}_{\Omega\Sigma\Xi}f_{\Lambda\Upsilon}{}^\Sigma + X^{(\rm CS)}_{\Xi\Sigma\Omega}f_{\Lambda\Upsilon}{}^\Sigma + X^{(\rm CS)}_{\Omega\Lambda\Sigma}f_{\Xi\Upsilon}{}^\Sigma\\
&\qquad + \frac{1}{2}X^{(\rm CS)}_{\Sigma\Lambda\Omega}f_{\Xi\Upsilon}{}^\Sigma\Big)\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}{}^\Lambda A_\rho{}^\Xi A_\sigma{}^\Upsilon\\
& -\frac{1}{8}\Lambda^\Omega\Big(X^{(\rm CS)}_{\Theta\Sigma\Delta}f_{\Omega\Lambda}{}^\Sigma + X^{(\rm CS)}_{\Sigma\Lambda\Delta}f_{\Omega\Theta}{}^\Sigma\\
&\qquad + X^{(\rm CS)}_{\Theta\Lambda\Sigma}f_{\Omega\Delta}{}^\Sigma\Big)f_{\Xi\Upsilon}{}^\Lambda\varepsilon^{\mu\nu\rho\sigma}A_\mu{}^\Xi A_\nu{}^\Upsilon A_\rho{}^\Theta A_\sigma{}^\Delta\Big]\,,
\end{aligned}$$

where we used the Jacobi identity (7.70) and the property (5.62).

A careful calculation finally shows that the supersymmetry variation of $S_{\rm CS}$ is

$$\begin{aligned}
\delta(\epsilon)S_{\rm CS} \;=\; & -\frac{1}{2}\int \mathrm{d}^4x\,\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Re}\Big[X^{(\rm CS)}_{\Omega\Lambda\Sigma}A_\mu{}^\Omega F_{\nu\rho}{}^\Lambda \qquad (5.66)\\
& + X^{(\rm CS)}_{[\Omega|\Lambda|\Sigma}f_{\Xi\Upsilon]}{}^\Lambda\, A_\mu{}^\Upsilon A_\nu{}^\Omega A_\rho{}^\Xi\Big]\bar\epsilon_{(L)}\gamma_\sigma\lambda^\Sigma_{(R)}\,.
\end{aligned}$$

## 5.4 Cancellation of gauge and supersymmetry anomalies

In this section, we combine the classical non-invariances of $(\hat S_f + S_{\rm CS})$ with the non-invariances induced by quantum anomalies.

### The consistent anomaly

The physical information of a quantum field theory is contained in the Green's functions, which in turn are encoded in an appropriate effective action denoted by $\Gamma[A]$. Recall from §2.4 that $\Gamma[A]$ can be written as a path integral over the other matter fields,

$$\mathrm{e}^{\mathrm{i}\Gamma[A]} = \int [d\psi d\bar{\psi}]\, \mathrm{e}^{\mathrm{i}S(A,\psi,\bar{\psi})} . \tag{5.67}$$

Even if the classical action, $S$, is gauge invariant, a non-invariance of the path integral measure may occur and violate the gauge invariance of $\Gamma[A]$, leading to a quantum anomaly:

$$\delta(\Lambda)\Gamma[A] \equiv \int \mathrm{d}^4x\, \Lambda^\Sigma \mathcal{A}_\Sigma . \tag{5.68}$$

Similarly there are supersymmetry anomalies, such that the final non-invariance of the one-loop effective action is

$$\mathcal{A} = \delta\Gamma[A] = \delta(\Lambda)\Gamma[A] + \delta(\epsilon)\Gamma[W] = \int \mathrm{d}^4x\, \left(\Lambda^\Sigma \mathcal{A}_\Sigma + \bar{\epsilon}\mathcal{A}_\epsilon\right) . \tag{5.69}$$

This anomaly should satisfy the Wess-Zumino consistency conditions [113], which guarantee that the variations (5.69) satisfy the symmetry algebra. E.g. for the gauge anomalies these are:

$$\delta(\Lambda_1)\left(\Lambda_2^\Sigma \mathcal{A}_\Sigma\right) - \delta(\Lambda_2)\left(\Lambda_1^\Sigma \mathcal{A}_\Sigma\right) = \Lambda_1^\Sigma \Lambda_2^\Omega f_{\Sigma\Omega}{}^\Lambda \mathcal{A}_\Lambda . \tag{5.70}$$

If the effective action is non-invariant under gauge transformations, then also its supersymmetry transformation is non-vanishing. As we explained before, this can for example be seen from the algebra (5.46).

A full cohomological analysis of anomalies in supergravity was made by F. Brandt in [97,98]. His result (see especially (9.2) in [98]) is that the total anomaly should be of the form (5.69) with

$$\begin{aligned}
\mathcal{A}_\Sigma &= -\frac{1}{4}\mathrm{i}\left[d_{\Lambda\Omega\Sigma}F_{\mu\nu}{}^\Omega + (d_{\Lambda\Omega\Xi}f_{\Sigma\Upsilon}{}^\Omega + \frac{3}{2}d_{\Lambda\Omega\Sigma}f_{\Xi\Upsilon}{}^\Omega)A_\mu{}^\Xi A_\nu{}^\Upsilon\right]\tilde{F}^{\mu\nu\Lambda} ,\\
\bar{\epsilon}\mathcal{A}_\epsilon &= \mathrm{Re}\left[\frac{3}{2}\mathrm{i}d_{\Lambda\Sigma\Omega}\bar{\epsilon}_{(R)}\lambda^\Omega_{(R)}\bar{\lambda}^\Lambda_{(L)}\lambda^\Sigma_{(L)} + \mathrm{i}d_{\Lambda\Sigma\Omega}A_\nu{}^\Omega\tilde{F}^{\mu\nu\Lambda}\bar{\epsilon}_{(L)}\gamma_\mu\lambda^\Sigma_{(R)}\right.\\
&\qquad \left. + \frac{3}{8}d_{\Lambda\Sigma\Omega}f_{\Xi\Upsilon}{}^\Lambda\varepsilon^{\mu\nu\rho\sigma}A_\mu{}^\Xi A_\nu{}^\Upsilon A_\sigma{}^\Omega\bar{\epsilon}_{(L)}\gamma_\rho\lambda^\Sigma_{(R)}\right] .
\end{aligned} \tag{5.71}$$

This result is true up to local counterterms, or equivalently, the anomalies have a scheme dependence. Choosing a different scheme is equivalent to the choice of another GCS term, i.e., a redefinition of $X^{(\mathrm{CS})}_{\Omega\Lambda\Sigma}$. As reviewed in [10] one can always choose a renormalization scheme in which the anomaly is proportional to $d_{\Lambda\Sigma\Omega}$. These constant coefficients form a totally symmetric tensor that is not fixed by the consistency conditions like (5.70). In the simplest case, the coefficients $d_{\Lambda\Sigma\Omega}$ are given by the symmetric trace of three gauge generators in the representation of the chiral fermions (see (2.74)).

### The cancellation

Since the anomaly $\mathcal{A}$ is a local polynomial in $A_\mu$, one might envisage a cancellation of the quantum anomaly by the classically non-gauge invariant terms in the action in the spirit of the Green-Schwarz mechanism.

The sum of the variations of the kinetic terms, (5.55) and (5.59), and of the variations of the GCS term, (5.65) and (5.67), simplifies if we set

$$X^{(\mathrm{CS})}_{\Omega\Lambda\Sigma} = X^{(\mathrm{m})}_{\Omega\Lambda\Sigma} = X_{\Omega\Lambda\Sigma} - X^{(s)}_{\Omega\Lambda\Sigma}\,, \tag{5.72}$$

and then use the consistency condition (4.51) for the tensor $X_{\Omega\Lambda\Sigma}$. The result is

$$\begin{aligned}
&\delta(\Lambda)\left(\hat{S}_f + S_{\mathrm{CS}}\right) = \\
&\qquad \frac{1}{4}\mathrm{i}\int \mathrm{d}^4x\, \Lambda^\Omega \Big[X^{(\mathrm{s})}_{\Omega\Lambda\Sigma} F_{\mu\nu}{}^\Sigma + \Big(X^{(\mathrm{s})}_{\Xi\Lambda\Sigma} f_{\Omega\Upsilon}{}^\Sigma + \frac{3}{2} X^{(\mathrm{s})}_{\Omega\Lambda\Sigma} f_{\Xi\Upsilon}{}^\Sigma\Big) A_\mu{}^\Xi A_\nu{}^\Upsilon\Big] \tilde{F}^{\mu\nu\Lambda}\,, \\
&\delta(\epsilon)\left(\hat{S}_f + S_{\mathrm{CS}}\right) = \\
&\qquad \int \mathrm{d}^4x\, \mathrm{Re}\Big[-\frac{3}{2}\mathrm{i} X^{(\mathrm{s})}_{\Omega\Lambda\Sigma} \bar{\epsilon}_{(R)} \lambda^\Omega_{(R)} \bar{\lambda}^\Lambda_{(L)} \lambda^\Sigma_{(L)} - \mathrm{i} X^{(\mathrm{s})}_{\Omega\Lambda\Sigma} A_\nu{}^\Omega \tilde{F}^{\mu\nu\Lambda} \bar{\epsilon}_{(L)} \gamma_\mu \lambda^\Sigma_{(R)} \\
&\qquad\qquad - \frac{3}{8} X^{(\mathrm{s})}_{\Omega\Lambda\Sigma} f_{\Xi\Upsilon}{}^\Lambda \varepsilon^{\mu\nu\rho\sigma} A_\mu{}^\Xi A_\nu{}^\Upsilon A_\sigma{}^\Omega \bar{\epsilon}_{(L)} \gamma_\rho \lambda^\Sigma_{(R)}\Big]\,.
\end{aligned} \tag{5.73}$$

The integrand of these expressions cancel the gauge and supersymmetry anomaly in (5.71) if we set

$$X^{(\mathrm{s})}_{\Omega\Lambda\Sigma} = d_{\Lambda\Sigma\Omega}\,. \tag{5.74}$$

Thus, if $X^{(\mathrm{m})}_{\Omega\Lambda\Sigma} = X^{(\mathrm{CS})}_{\Omega\Lambda\Sigma}$ and $X^{(\mathrm{s})}_{\Omega\Lambda\Sigma} = d_{\Lambda\Sigma\Omega}$, both gauge and supersymmetry are unbroken, in particular anomaly-free. Note that this does not mean that any anomaly proportional to some $d_{\Lambda\Sigma\Omega}$ can be canceled by a $X^{(\mathrm{s})}_{\Omega\Lambda\Sigma}$. A gauge kinetic function with an appropriate gauge transformation induced by gauge transformations of scalar fields such that (5.74) holds may simply not exist. Our analysis only shows that *if* (5.74) holds, and $X^{(\mathrm{m})}_{\Omega\Lambda\Sigma} = X^{(\mathrm{CS})}_{\Omega\Lambda\Sigma}$ is satisfied, the theory is gauge and supersymmetry invariant. In the next section these conditions will be investigated in more detail for some particular choices of gauge groups. We will show that in the case of a semi-simple gauge group, these conditions are never satisfied, and one needs at least an Abelian $U(1)$ factor.

## 5.5 Specializing to Abelian $\times$ semi-simple gauge groups

In the previous sections we have described how an appropriate combination of Peccei-Quinn terms, generalized Chern-Simons terms and quantum anomalies can yield a gauge invariant and supersymmetric theory, even though each of these three contributions individually violate gauge and supersymmetry invariance. The criteria for a successful cancellation are given in terms of two conditions, (5.72) and (5.74), on the tensors $X_{\Omega\Lambda\Sigma}$, $X^{(\mathrm{CS})}_{\Omega\Lambda\Sigma}$ and $d_{\Lambda\Sigma\Omega}$. These conditions put strong constraints on different aspects of the theory, such as the anomalous fermionic spectrum, the gauge transformations, the form of the gauge kinetic function and the Chern-Simons terms.

In particular, one can show that these conditions are only satisfied for certain types of gauge groups. In fact, it is easy to verify that semi-simple gauge groups do not fall into this category. First, we recall that for semi-simple algebras the GCS terms do not bring anything new [112], at least in the classical theory. By this we mean they can be replaced by a redefinition of the kinetic matrix $f_{\Lambda\Sigma}$. Indeed, for semi-simple algebras one can always construct a constant real symmetric matrix $Z_{\Lambda\Sigma}$ such that

$$X^{(\mathrm{CS})}_{\Omega\Lambda\Sigma} = 2 f_{\Omega(\Lambda}{}^{\Xi} Z_{\Sigma)\Xi}\,, \tag{5.75}$$

for any tensor $X^{(\mathrm{CS})}_{\Omega\Lambda\Sigma}$. Then the action $S_{\mathrm{CS}}$ can be reabsorbed in the original action $S_f$ using

$$f'_{\Lambda\Sigma} = f_{\Lambda\Sigma} + \mathrm{i} Z_{\Lambda\Sigma}\,. \tag{5.76}$$

For a proof we refer to [112] or [1]. Without loss of generality, we can thus, for semi-simple gauge groups, think of $S_{\mathrm{CS}}$ as being absorbed into the kinetic term and set $X^{(\mathrm{CS})}_{\Omega\Lambda\Sigma} = 0$. The first condition (5.72) then implies that also $X^{(\mathrm{m})}_{\Omega\Lambda\Sigma} = 0$ in the resulting theory without Chern-Simons terms, and the second constraint (5.74) reduces to $X_{\Omega\Lambda\Sigma} = X^{(\mathrm{s})}_{\Omega\Lambda\Sigma} = d_{\Lambda\Sigma\Omega}$. Using the latter equality and the consistency condition (4.51), one can show [114] that

$$d_{\Lambda\Sigma\Xi} f_{\Omega\Upsilon}{}^{\Xi} = 0\,. \tag{5.77}$$

As a semi-simple group has no Abelian ideals, this equation implies that $d_{\Lambda\Sigma\Xi} = 0$. Therefore, anomaly freedom for theories with a semi-simple gauge group can only be achieved if the anomalies themselves vanish.

Supported by this result, we conclude that the simplest example to which our Green-Schwarz mechanism can be applied, is the product of a (one-dimensional) Abelian factor and a semi-simple gauge group. This example is particularly relevant for phenomenological reasons, and it has been considered before in the literature [10, 87, 88]. In order to clarify the relation between this work and our results, we will now specify to a $U(1) \times G$ gauge group, where $G$ is semi-simple. In this case, one can look at "mixed" anomalies, which are the ones proportional to

$d_{0ab} = \mathrm{Tr}(QT_aT_b)$, where $Q$ is the U(1) charge operator and $T_a$ are the generators of the semi-simple algebra. Following [88, section 2.2], one can add counterterms such that the mixed anomalies proportional to $\Lambda^a$ cancel and one remains with those that are of the form $\Lambda^0\,\mathrm{Tr}\left(Q\mathcal{G}_{\mu\nu}\tilde{\mathcal{G}}^{\mu\nu}\right)$, where $\Lambda^0$ is the Abelian gauge parameter and $\mathcal{G}_{\mu\nu} = \mathcal{G}^a_{\mu\nu}T_a$ the semi-simple field strength. Schematically, it looks like

$$\begin{array}{c|ccc}
\text{Anomalies:} & \Lambda^a\mathcal{A}^a_{\text{mixed con}} & + & \Lambda^0\mathcal{A}^0_{\text{mixed con}} \\
\hline
\delta(\Lambda)\mathcal{L}_{\text{ct}}: & -\Lambda^a\mathcal{A}^a_{\text{mixed con}} & - & \Lambda^0\mathcal{A}^0_{\text{mixed con}} \\
 & & + & \Lambda^0\mathcal{A}^0_{\text{mixed cov}} \\
\hline\hline
\text{sum:} & 0 & + & \Lambda^0\mathcal{A}^0_{\text{mixed cov}}
\end{array} \tag{5.78}$$

where the subscripts "con" and "cov" denote the consistent and covariant anomalies[13], respectively. The counterterms $\mathcal{L}_{\text{ct}}$ have the following form:

$$\mathcal{L}_{\text{ct}} = \frac{1}{3}Z\varepsilon^{\mu\nu\rho\sigma}C_\mu\,\mathrm{Tr}\left[Q\left(A_\nu\partial_\rho A_\sigma + \frac{3}{4}A_\nu A_\rho A_\sigma\right)\right], \qquad Z = \frac{1}{4\pi^2}, \tag{5.79}$$

where $C_\mu$ and $A_\mu$ are the gauge fields for the Abelian and semi-simple gauge groups respectively. The expressions for the anomalies are:

$$\begin{aligned}
\mathcal{A}^a_{\text{mixed con}} &= -\frac{1}{3}Z\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left[T^aQ\partial_\mu\left(C_\nu\partial_\rho A_\sigma + \frac{1}{4}C_\nu A_\rho A_\sigma\right)\right], \\
\mathcal{A}^0_{\text{mixed con}} &= -\frac{1}{6}Z\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left[Q\partial_\mu\left(A_\nu\partial_\rho A_\sigma + \frac{1}{2}A_\nu A_\rho A_\sigma\right)\right], \\
\mathcal{A}^0_{\text{mixed cov}} &= -\frac{1}{8}\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left[Q\mathcal{G}_{\mu\nu}\mathcal{G}_{\rho\sigma}\right].
\end{aligned} \tag{5.80}$$

The remaining anomaly $\mathcal{A}^0_{\text{mixed cov}}$ is typically canceled by the Green-Schwarz mechanism.

We will compare this now with our results for general non-Abelian gauge groups, which we reduce to the case Abelian × semi-simple. The indices $\Lambda, \Sigma, \ldots$ are split into 0 for the U(1) and $a$ for the semi-simple group generators. We expect the GCS terms (5.61) to be equivalent to the counterterms in [88] and the role of the Green-Schwarz mechanism is played by a U(1) variation of the kinetic terms $f_{ab}$, hence by a $X$-tensor with non-trivial components $X_{0ab}$.

It follows from the consistency condition (4.51) that

$$X_{00a} = X_{a00} = 0\,, \tag{5.81}$$

[13] The form of the anomaly that satisfies the Wess-Zumino consistency condition (5.70) is known as the consistent anomaly. This is the only physically relevant form, since it follows from the variation of the effective action. The covariant anomaly on the other hand, is a quadratic expression in the covariant field strengths, and in some cases it is related to the consistent anomaly via local counterterms. This is true, for example, for the mixed anomaly $\mathcal{A}^a_{\text{mixed cov}}$, as is clear from (5.78). For more details, we refer to [88, 115].

and the $X_{0ab}$'s are proportional to the Cartan-Killing metric in each simple factor. We write here

$$X_{0ab} = Z\,\mathrm{Tr}(QT_aT_b)\,, \tag{5.82}$$

where $Z$ could be arbitrary, but our results will match the results of [88] for the value of $Z$ in (5.79).

We will not allow for off-diagonal elements of the gauge kinetic function $f_{\Lambda\Sigma}$:

$$f_{0a} = 0 \quad \Rightarrow \quad X_{b0a} = 0\,. \tag{5.83}$$

There may be non-zero components $X_{000}$ and $X_{abc}$, but we shall be concerned here only with the mixed ones, i.e., we have only (5.82) different from zero.

If we reduce (5.55) using (5.82) we get

$$\left[\delta(\Lambda)\hat{S}_f\right]_{\mathrm{mixed}} = \int \mathrm{d}^4x \left[\frac{1}{8} Z\Lambda^0 \varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left(Q\mathcal{G}_{\mu\nu}\mathcal{G}_{\rho\sigma}\right)\right]. \tag{5.84}$$

Splitting (5.82) into a totally symmetric and mixed symmetry part gives

$$X^{(\mathrm{s})}_{0ab} = X^{(\mathrm{s})}_{b0a} = \frac{1}{3}X_{0ab} = \frac{1}{3}Z\,\mathrm{Tr}(QT_aT_b)\,, \tag{5.85}$$

$$X^{(\mathrm{m})}_{0ab} = \frac{2}{3}X_{0ab} = \frac{2}{3}Z\,\mathrm{Tr}(QT_aT_b)\,, \qquad X^{(\mathrm{m})}_{b0a} = -\frac{1}{3}X_{0ab} = -\frac{1}{3}Z\,\mathrm{Tr}(QT_aT_b)\,.$$

We learned in §5.4 that for a final gauge and supersymmetry invariant theory we have to take $X^{\mathrm{CS}} = X^{(\mathrm{m})}$, and hence the mixed part of the GCS action (5.61) reads in this case:

$$[S_{\mathrm{CS}}]_{\mathrm{mixed}} = \int \mathrm{d}^4x \left[\frac{1}{3} ZC_\mu \varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left(QA_\nu\partial_\rho A_\sigma + \frac{3}{4}A_\nu A_\rho A_\sigma\right)\right]. \tag{5.86}$$

Finally, we reduce the consistent anomaly (5.71) using $d_{\Lambda\Sigma\Omega} = X^{(\mathrm{s})}_{\Omega\Lambda\Sigma}$. We find

$$\begin{aligned}
\mathcal{A}_0 &= -\frac{1}{6}Z\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left[Q\partial_\mu\left(A_\nu\partial_\rho A_\sigma + \frac{1}{2}A_\nu A_\rho A_\sigma\right)\right],\\
\mathcal{A}_a &= -\frac{1}{3}Z\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left[T_aQ\partial_\mu\left(C_\nu\partial_\rho A_\sigma + \frac{1}{4}C_\nu A_\rho A_\sigma\right)\right],
\end{aligned} \tag{5.87}$$

where $G_{\mu\nu}$ is the Abelian part of the gauge field $\mathcal{G}_{\mu\nu}$.

We can make the following observations:

1. The mixed part of the GCS action (5.86) is indeed equal to the counterterms (5.79), introduced in [88].

2. The consistent anomalies (5.87), for which we based our formula on [97, 98], match those in the first two lines of (5.80). As we mentioned above, the counterterm has modified the resulting anomaly to the covariant form in the last line of (5.80).

3. We see that the variation of the kinetic term for the vector fields (5.84) is able to cancel this mixed covariant anomaly (this is the Green-Schwarz mechanism).

Combining these remarks, our cancellation procedure can schematically be presented as follows:

$$\begin{array}{r|ccc}
\text{Anomalies:} & \Lambda^a\mathcal{A}^a_{\text{mixed con}} & + & \Lambda^0\mathcal{A}^0_{\text{mixed con}} \\
\hline
\delta(\Lambda)\mathcal{L}_{(\text{CS})}: & -\Lambda^a\mathcal{A}^a_{\text{mixed con}} & - & \Lambda^0\mathcal{A}^0_{\text{mixed con}} \\
 & & + & \Lambda^0\mathcal{A}^0_{\text{mixed cov}} \\
\hline
\delta(\Lambda)\hat{S}_f: & & - & \Lambda^0\mathcal{A}^0_{\text{mixed cov}} \\
\hline\hline
\text{sum:} & 0 & + & 0
\end{array} \tag{5.88}$$

## 5.6 Supergravity corrections

In this section, we generalize our treatment to the full $4D$, $\mathcal{N}=1$ supergravity theory. We check supersymmetry and gauge invariance of the supergravity action and show that no extra GCS terms (besides those already added in the rigid theory) have to be included to obtain supersymmetry or gauge invariance.

The simplest way to go from rigid supersymmetry to supergravity makes use of the superconformal tensor calculus [35, 37–39]. A summary in this context is given in [9]. Compared to the rigid theory, the additional fields reside in the conformal supergravity multiplet[14], i.e., the gauge multiplet of the superconformal algebra, and a compensating multiplet. The Weyl multiplet contains the vierbein, the gravitino $\Psi_\mu$ and an auxiliary vector, which will not be important for us. The compensating multiplet enlarges the set of chiral multiplets in the theory by one. The full set of fields in the chiral multiplets is now ($X^I$, $\Omega^I$, $H^I$), which denote complex scalars, fermions and complex auxiliary fields, respectively. The physical chiral multiplets $(z^i, \chi^i, h^i)$ form a subset of these such that $I$ runs over one more value than $i$. As our final results depend only on the vector multiplet, this addition will not be very important for us, and we do not have to discuss how the physical ones are embedded in the full set of chiral multiplets.

When going from global supersymmetry to supergravity, several changes to the action (5.50) are required. As we pointed out already in §3.3, extra terms appear

[14] In the literature this is often called the Weyl multiplet, not to be confused with the multiplet $W_{\mu\nu\alpha}$ that we introduced in (5.17).

that are proportional to the gravitino $\Psi_\mu$, and one has to add an extra factor $e$, which is the determinant of the vierbein. All these changes are taken into account if one replaces the integrand of (5.50) by the so-called density formula, which is rather simple due to the use of the superconformal calculus [33]:

$$S_f = \int \mathrm{d}^4x\, e\, \mathrm{Re}\left[h(fW^2) + \bar{\Psi}_{\mu(R)}\gamma^\mu \chi_{(L)}(fW^2) + \frac{1}{2}\bar{\Psi}_{\mu(R)}\gamma^{\mu\nu}\Psi_{\nu(R)} z(fW^2)\right], \tag{5.89}$$

For completeness, we give the component expression of (5.89). It can be found by plugging in the relations (5.49), where we replace the fields of the chiral multiplets with an index $i$ by the larger set indexed by $I$, into the density formula (5.89). The result is

$$\begin{aligned} \hat{S}_f \;=\; \int \mathrm{d}^4x\, e\Big[ &\mathrm{Re}\, f_{\Lambda\Sigma}(X)\left(-\frac{1}{4}\mathcal{F}_{\mu\nu}{}^{\Lambda}\mathcal{F}^{\mu\nu\,\Sigma} - \frac{1}{2}\bar{\lambda}^\Lambda\gamma^\mu\hat{D}_\mu\lambda^\Sigma + \frac{1}{2}D^\Lambda D^\Sigma \right.\\ &\left. + \frac{1}{8}\bar{\Psi}_\mu\gamma^{\nu\rho}\left(\mathcal{F}_{\nu\rho}{}^{\Lambda} + \hat{\mathcal{F}}_{\nu\rho}{}^{\Lambda}\right)\gamma^\mu\lambda^\Sigma\right)\\ &+ \frac{1}{4}\mathrm{i}\,\mathrm{Im}\, f_{\Lambda\Sigma}(X)\mathcal{F}_{\mu\nu}{}^{\Lambda}\tilde{\mathcal{F}}^{\mu\nu\Sigma} + \frac{1}{4}\mathrm{i}\left(\hat{D}_\mu\,\mathrm{Im}\, f_{\Lambda\Sigma}(X)\right)\bar{\lambda}^\Lambda\gamma_5\gamma^\mu\lambda^\Sigma\\ &+ \left\{\frac{1}{2}\partial_I f_{\Lambda\Sigma}(X)\left[\bar{\Omega}^I_{(L)}\left(-\frac{1}{2}\gamma^{\mu\nu}\hat{\mathcal{F}}_{\mu\nu}{}^{\Lambda} + \mathrm{i}D^\Lambda\right)\lambda^\Sigma_{(L)}\right.\right.\\ &\left. - \frac{1}{2}\left(H^I + \bar{\Psi}_{\mu(R)}\gamma^\mu\Omega^I_{(L)}\right)\bar{\lambda}^\Lambda_{(L)}\lambda^\Sigma_{(L)}\right]\\ &\left. + \frac{1}{4}\partial_I\partial_J f_{\Lambda\Sigma}(X)\,\bar{\Omega}^I_{(L)}\Omega^J_{(L)}\bar{\lambda}^\Lambda_{(L)}\lambda^\Sigma_{(L)} + \mathrm{h.c.}\right\}\Big], \end{aligned} \tag{5.90}$$

where the hat denotes full covariantization with respect to gauge and local supersymmetry, e.g.

$$\hat{\mathcal{F}}_{\mu\nu}{}^{\Sigma} = \mathcal{F}_{\mu\nu}{}^{\Sigma} + \bar{\Psi}_{[\mu}\gamma_{\nu]}\lambda^\Sigma\,. \tag{5.91}$$

Note that we use already the derivative $D_\mu\,\mathrm{Im}\, f_{\Lambda\Sigma}(X)$, covariant with respect to the shift symmetries, as explained around (5.53). Therefore, we denote this action as $\hat{S}_f$ as we did for rigid supersymmetry.

The kinetic matrix $f_{\Lambda\Sigma}$ is now a function of the scalars $X^I$. We thus have in the superconformal formulation

$$\delta_\Omega f_{\Lambda\Sigma} = \partial_I f_{\Lambda\Sigma}\delta_\Omega X^I = \mathrm{i}X_{\Omega\Lambda\Sigma} + \ldots\,. \tag{5.92}$$

Let us first consider the supersymmetry variation of (5.90). Similar to (5.59), the variation of (5.90) can only get non-vanishing contributions that are proportional to the $X$-tensor, because for $X_{\Omega\Lambda\Sigma}=0$, the action $\hat{S}_f$ is invariant under all local superconformal transformations. The non-vanishing terms are of two different kinds. First, we obtain the same expression as in (5.59) because the sources of these non-invariances are still present in (5.90). Second, there can be extra contributions – proportional to the gravitino $\Psi_\mu$ – that come from the variation of $H^I$ and $\Omega^I$ in covariant objects that are now also covariantized with respect to the supersymmetry transformations and from the variation of $e$ and $\lambda^\Sigma$ in the gauge covariantization of the $(\hat{D}_\mu\,\mathrm{Im}\, f_{\Lambda\Sigma})$-term. Let us list in more detail the parts of the action that give these extra contributions.

First there is a coupling of $\Omega^I$ with a gravitino and gaugini, coming from $-\frac{1}{4}e\partial_I f_{\Lambda\Sigma}\bar{\Omega}^I_{(L)}\gamma^{\mu\nu}\hat{\mathcal{F}}_{\mu\nu}{}^{\Lambda}\lambda^{\Sigma}_{(L)}$:

$$S_1=\int \mathrm{d}^4x\, e\Big[-\frac{1}{4}\partial_I f_{\Lambda\Sigma}\bar{\Omega}^I_{(L)}\gamma^{\mu\nu}\lambda^{\Sigma}_{(L)}\bar{\Psi}_{[\mu}\gamma_{\nu]}\lambda^{\Lambda}+\text{h.c.}\Big] \tag{5.93}$$

$$\rightarrow \delta(\epsilon)S_1=\int \mathrm{d}^4x\, e\Big[-\frac{1}{8}\mathrm{i}X_{\Omega\Lambda\Sigma}A_\rho{}^{\Omega}\bar{\lambda}^{\Sigma}_{(L)}\gamma^{\mu\nu}\gamma^{\rho}\epsilon_{(R)}\bar{\Psi}_\mu\gamma_\nu\lambda^{\Lambda}+\ldots+\text{h.c.}\Big].$$

We used the expression (5.91) for $\hat{\mathcal{F}}^A_{\mu\nu}$ and (5.47) where $D_\mu X^I$ is now also covariantized with respect to the supersymmetry transformations, i.e. $\hat{D}_\mu X^I$. There is another coupling between $\Omega^I$, a gravitino and gaugini that we will treat separately:

$$S_2=\int \mathrm{d}^4x\, e\Big[\frac{1}{4}\partial_I f_{\Lambda\Sigma}\bar{\Omega}^I_{(L)}\gamma^{\mu}\Psi_{\mu(R)}\bar{\lambda}^{\Lambda}_{(L)}\lambda^{\Sigma}_{(L)}+\text{h.c.}\Big] \tag{5.94}$$

$$\rightarrow \delta(\epsilon)S_2=\int \mathrm{d}^4x\, e\Big[\frac{1}{8}\mathrm{i}X_{\Omega\Lambda\Sigma}A_\rho{}^{\Omega}\bar{\epsilon}_{(R)}\gamma^{\rho}\gamma^{\mu}\Psi_{\mu(R)}\bar{\lambda}^{\Lambda}_{(L)}\lambda^{\Sigma}_{(L)}+\ldots+\text{h.c.}\Big].$$

A third contribution comes from the variation of the auxiliary field $H^I$ in $S_3$, where

$$S_3=\int \mathrm{d}^4x\, e\Big[-\frac{1}{4}\partial_I f_{\Lambda\Sigma}H^I\bar{\lambda}^{\Lambda}_{(L)}\lambda^{\Sigma}_{(L)}+\text{h.c.}\Big]. \tag{5.95}$$

The variation is of the form

$$\begin{aligned}\delta(\epsilon)H^I &= \bar{\epsilon}_{(R)}\gamma^\mu D_\mu\Omega^I_{(L)}+\ldots=-\frac{1}{2}\bar{\epsilon}_{(R)}\gamma^\mu\gamma^\nu\hat{D}_\nu X^I\Psi_{\mu(R)}+\ldots\\ &= \frac{1}{2}\delta_\Omega X^I A_\nu{}^{\Omega}\bar{\epsilon}_{(R)}\gamma^\mu\gamma^\nu\Psi_{\mu(R)}+\ldots\,.\end{aligned} \tag{5.96}$$

Therefore we obtain

$$S_3=\int \mathrm{d}^4x\, e\Big[-\frac{1}{4}\partial_I f_{\Lambda\Sigma}H^I\bar{\lambda}^{\Lambda}_{(L)}\lambda^{\Sigma}_{(L)}+\text{h.c.}\Big] \tag{5.97}$$

$$\rightarrow \delta(\epsilon)S_3=\int \mathrm{d}^4x\, e\Big[-\frac{1}{8}\mathrm{i}X_{\Omega\Lambda\Sigma}A_\nu{}^{\Omega}\bar{\epsilon}_{(R)}\gamma^\mu\gamma^\nu\Psi_{\mu(R)}\bar{\lambda}^{\Lambda}_{(L)}\lambda^{\Sigma}_{(L)}+\ldots+\text{h.c.}\Big].$$

Finally, we need to consider the variation of the vierbein $e$ and the gaugini in a part of the covariant derivative on $\operatorname{Im} f_{\Lambda\Sigma}$:

$$S_4 = \int \mathrm{d}^4x\, e\left[\frac{1}{4}\mathrm{i}X_{\Omega\Lambda\Sigma}A_\mu{}^\Omega\bar{\lambda}^\Lambda\gamma^\mu\gamma_5\lambda^\Sigma\right] \tag{5.98}$$

$$\begin{aligned}\rightarrow \delta(\epsilon)S_4 = \int \mathrm{d}^4x\, e\Big[ &-\frac{1}{4}\mathrm{i}X_{\Omega\Lambda\Sigma}A_\rho{}^\Omega\Big(\bar{\lambda}^\Lambda_{(R)}\gamma^\mu\lambda^\Sigma_{(L)}\bar{\epsilon}_{(R)}\gamma^\rho\Psi_{\mu(L)}\\ &\qquad+\frac{1}{4}\bar{\epsilon}_{(R)}\gamma^\rho\gamma^\mu\gamma^\nu\Psi_{\nu(L)}\bar{\lambda}^\Lambda_{(L)}\gamma_\mu\lambda^\Sigma_{(R)}\\ &\qquad+\frac{1}{4}\bar{\epsilon}_{(R)}\gamma^\rho\gamma^\mu\Psi_{\mu(R)}\bar{\lambda}^\Lambda_{(L)}\lambda^\Sigma_{(L)}\Big)\\ &+\frac{1}{4}\mathrm{i}X_{\Omega\Sigma\Lambda}A^{\mu\Omega}\bar{\Psi}_{\mu(R)}\epsilon_{(R)}\bar{\lambda}^\Lambda_{(L)}\lambda^\Sigma_{(L)}+\ldots+\mathrm{h.c.}\Big]\,.\end{aligned}$$

It requires some careful manipulations to obtain the given result for $\delta(\epsilon)S_4$. One needs the variation of the determinant of the vierbein, gamma matrix identities and Fierz relations.

In the end, we find that $\delta(\epsilon)\,(S_1 + S_2 + S_3 + S_4) = 0$. This means that all extra contributions that were not present in the supersymmetry variation of the original supergravity action vanish without the need of extra terms (e.g. generalizations of the GCS terms). We should also remark here that the variation of the GCS terms themselves is not influenced by the transition from global supersymmetry to supergravity because it depends only on the vectors $A_\mu^\Lambda$, whose supersymmetry transformations have no gravitino corrections in $\mathcal{N} = 1$.

Let us check now the gauge invariance of terms proportional to the gravitino. Neither terms involving the real part of the gauge kinetic function, $\operatorname{Re} f_{\Lambda\Sigma}$, nor its derivatives violate the gauge invariance of $\hat{S}_f$. The only contributions to gauge non-invariance come from the pure imaginary parts, $\operatorname{Im} f_{\Lambda\Sigma}$, of the gauge kinetic function. On the other hand, no extra $\operatorname{Im} f_{\Lambda\Sigma}$ terms appear when one goes from global supersymmetry to supergravity and, hence, we do not have any extra contributions to the gauge variation of the full supergravity action, apart from those that were already present in the rigid case. This is consistent with our earlier result that neither $\delta(\epsilon)\hat{S}_f$ nor $S_{\mathrm{CS}}$ contain gravitini.

Consequently, the general $\mathcal{N} = 1$ supergravity action contains just the extra terms (5.61), and we can add them to the original action in [8].

## 5.7 Cancellation of mixed gauge - gravitational anomalies

So far we have only considered the cancellation of pure gauge anomalies. However, field theories with chiral matter coupled to gravity, such as $4D$, $\mathcal{N} = 1$ supergravity, are also plagued by mixed gauge-gravitational anomalies [99]. It is the purpose of this section to construct an extension of the 4-dimensional GS mechanism and incorporate the possibility of mixed anomaly cancellation.

In §5.2 we studied a simple example from which we could identify the relevant ingredients that form the basis of this mechanism. In particular, we found evidence for higher order couplings (proportional to the Hirzebruck signature density) that play the role of counterterms. Here, we will investigate the precise form of these counterterms in more detail, as well as their appropriate transformation and the subsequent conditions for mixed anomaly cancellation. But before we do that, let us specify the correct form of the gravitational anomaly.

### Mixed anomalies

It is a well known fact that pure gravitational anomalies do not exist in 4 dimensions, see [99] or recall footnote 2 in the introduction of this chapter. However, triangle one-loop diagrams with chiral fermions running in the loop and two energy momentum tensors and a U(1) gauge current at the external legs can be anomalous. This anomaly is called the *mixed gauge-gravitational anomaly* for obvious reasons. Let us review how it also arises as a gauge or local Lorentz variation of the effective action.

In general, we have

$$(\delta(\Lambda) + \delta(\lambda))\, \Gamma[A, \omega] = \int \mathrm{d}^4x\, I_{4,1}\,, \tag{5.99}$$

where $\Lambda^{\Sigma}(x)$ and $\lambda^{ab}(x)$ are the gauge and local Lorentz parameters respectively and $A_{\mu}{}^{\Sigma}$ and $\omega_{\mu}{}^{ab}$ are the gauge fields and spin connection. We denoted the effective action with $\Gamma[A, \omega]$ and $I_{4,1}$ (which is nothing but the anomaly) is given by the descent equations:

$$\begin{aligned} I_6 &= \mathrm{d}I_5, \\ (\delta(\Lambda) + \delta(\lambda))\, I_5 &= \mathrm{d}I_{4,1}. \end{aligned} \tag{5.100}$$

The subscript $m$ in $I_{m,n}$ indicates that it is a $m$-form whereas the index $n$ indicates that the expression is $n$th order in the parameters $\Lambda^{\Sigma}$ and $\lambda^{ab}$. Furthermore, from the analysis in [24, 99] we know that $I_6$ can be written as

$$I_6 = \frac{1}{24\pi^2}\mathrm{Tr}F^3 + \frac{1}{192\pi^2}\mathrm{Tr}F\,\mathrm{tr}R^2\,, \tag{5.101}$$

where we used again Tr and tr to denote a trace in the adjoint and vector representations respectively, i.e.,

$$\mathrm{tr}R^2 = \frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}R_{\mu\nu ab}R_{\rho\sigma}{}^{ab}\,, \tag{5.102}$$

which is the Hirzebruck signature density. The first term in equation (5.101) descends to the familiar expression (2.73) for pure gauge anomalies. The second term represents the mixed gauge-gravitational anomaly. Since for any simple Lie algebra, $\mathrm{Tr}F = 0$, only $U(1)$ parts can contribute, in which case

$$\mathrm{Tr}F = \sum_{\ell,s} q_\ell^{(s)} F^{(s)}, \tag{5.103}$$

where the superscript $s$ labels different U(1)-factors and $\ell$ runs over the different right-handed chiral fermions that are charged under $\mathrm{U}(1)_s$.

Next we can apply the descent equations on $I_6$ to obtain the correct form of the anomaly. There is no unique way to do the descent, which corresponds to the possibility to change the form of the anomaly by adding a local counterterm to the effective action (without being able to remove a relevant anomaly altogether). The different ways to do the descent will either lead to an effective action that is not gauge invariant or to one that is not local Lorentz invariant. The local counterterm allows one to interpolate between both possibilities. Let us first present both effective actions:

$$\delta(\Lambda)\Gamma^{(1)} = \frac{1}{192\pi^2}\int\left(\sum_{\ell,s} q_\ell^{(s)}\Lambda^{(s)}\right)\mathrm{tr}R^2, \qquad \delta(\lambda)\Gamma^{(2)} = 0, \tag{5.104}$$

with $\Lambda^{(s)}$ the gauge parameters for each $U(1)_s$ factor: $\delta(\Lambda)A_\mu^{(s)} = \partial_\mu\Lambda^{(s)}$, or

$$\delta(\lambda)\Gamma^{(2)} = \frac{1}{192\pi^2}\int\left(\sum_{\ell,s} q_\ell^{(s)}F^{(s)}\right)\mathrm{tr}\lambda\mathrm{d}\omega, \qquad \delta(\Lambda)\Gamma^{(1)} = 0, \tag{5.105}$$

where $\omega_\mu{}^{ab}$ (not to be confused with the Chern-Simons three-forms $\omega_3$) transforms as in (3.18): $\delta(\lambda)\omega_\mu{}^{ab} = \partial_\mu\lambda^{ab} - 2\lambda_c{}^{[a}\omega_\mu{}^{b]c}$.

Clearly, $\Gamma^{(1)}$ is invariant under local Lorentz transformations but transforms non-trivially under gauge transformations, and vice versa for $\Gamma^{(2)}$. We can interpolate between the two by adding a local counterterm to the effective action:

$$\Delta\Gamma = \frac{1}{192\pi^2}\int\left(\sum_{\ell,s} q_\ell^{(s)}A^{(s)}\right)\mathrm{tr}\,\omega_3^{(\mathrm{L})}, \tag{5.106}$$

such that

$$\delta(\Lambda)\Gamma^{(1)} + (\delta(\Lambda) + \delta(\lambda))\,\Delta\Gamma = \delta(\lambda)\Gamma^{(2)}\,. \tag{5.107}$$

In the remainder of this text, we will use $\Gamma^{(1)}$ as the effective action and the right hand side of (5.104) as the corresponding anomaly.

### Higher derivative corrections to $4D$, $\mathcal{N}=1$ supergravity

The mixed gauge-gravitational anomaly in (5.104) is a 4th order expression in the space-time derivatives. Therefore, a Green-Schwarz cancellation mechanism necessarily involves the addition of higher order derivative corrections to the original supergravity action. Moreover, these "counterterms" should have the correct gauge transformation in order to cancel the anomaly. In the context of our simple example in §5.2, we found that $S_{\rm Weyl}$ in (5.17) and the shift transformation in (5.22) do the job. This result will now be explored in a more general context, i.e., we start from the following generic superfield expression for the higher derivative terms:

$$\mathcal{L}_{\rm Weyl} = -\int \mathrm{d}^2\theta\,\mathcal{E}\,\left(f(S)W_{\mu\nu\alpha}W^{\mu\nu}_{\beta}\varepsilon^{\alpha\beta}\right) + \text{h.c.}\,, \tag{5.108}$$

where $f$ is an arbitrary holomorphic function of the scalar fields that are the lowest components of the chiral superfields $S^i$, $i = 1,\ldots,n_C$.

In order to make our discussion as transparent as possible, we will first write down a component expression for $\mathcal{L}_{\rm Weyl}$. Such an expression was already constructed a long time ago [96], but it is good to reconsider the different steps in the calculation. From the linearized results in [116] it can be seen that the lowest component of the chiral superfield $W_{\mu\nu\alpha}$ is the gravitino field strength $R_{\mu\nu(L)}(Q)$:

$$R_{\mu\nu(L)}(Q) = 2\left(\partial_{[\mu} + \frac{1}{4}\omega_{[\mu}{}^{ab}\gamma_{ab}\right)\Psi_{\nu](L)}\,. \tag{5.109}$$

Using supersymmetry, one should be able to calculate also the other components of $W_{\mu\nu\alpha}$ and plug the result into (5.30).

However, we can also follow a different (and easier) approach, which is superconformal tensor calculus [32, 33, 35]. This formalism allows us to construct an action for the Weyl multiplet (coupled to scalars via the holomorphic function $f$) that is invariant under all superconformal transformations. We start from the observation that $fW_{\mu\nu\alpha}W^{\mu\nu}_{\beta}\varepsilon^{\alpha\beta}$ is a chiral multiplet and therefore we can use the superconformal density formula (5.89). Once the superconformal extension of $\mathcal{L}_{\rm Weyl}$ is known, an appropriate gauge fixing procedure brings us back to the desired Poincaré theory. We should note that our treatment is not self-contained; we will make use of several results in superconformal tensor calculus that will not be repeated here, but can be found in the literature, see e.g. [35, 117].

We start this program by constructing the components of the superconformal analogue of the Poincaré Weyl multiplet. The lowest component $R_{\mu\nu(L)}(Q)$ should be replaced by its superconformal covariant counterpart, denoted by $\widehat{R}_{\mu\nu(L)}(Q)$:

$$\widehat{R}_{\mu\nu(L)}(Q) = 2\left(\partial_{[\mu} + \frac{1}{4}\omega_{[\mu}{}^{ab}\gamma_{ab} + \frac{1}{2}b_{[\mu} + \frac{1}{2}\mathrm{i}\gamma_5\mathcal{A}_{\mu]}\right)\Psi_{\nu](L)} - 2\gamma_{[\mu}\phi_{\nu](L)}\,. \quad (5.110)$$

Since this is the lowest component of a chiral multiplet, the other components can be determined by acting on $\widehat{R}_{\mu\nu(L)}(Q)$ with a supersymmetry transformation. In superconformal tensor calculus, there exists a short-cut, see [35], that allows us to obtain the result in a straightforward way. The supersymmetry transformation of an arbitrary superconformal curvature is given by

$$\delta(\epsilon)\widehat{R}_{ab}{}^{A} = \epsilon^{B}\widehat{R}_{ab}{}^{C}f_{CB}{}^{A} + 2\epsilon^{B}D_{[a}\mathcal{M}_{b]B}{}^{A} - 2\epsilon^{C}\mathcal{M}_{[aC}{}^{B}\mathcal{M}_{b]B}{}^{A}\,, \quad (5.111)$$

where $A, B, \ldots$ label superconformal transformations, the $\mathcal{M}_{\mu A}{}^{B}$ are the matter parts in the superconformal transformation of the gauge fields $h_\mu{}^B$, the $f_{CB}{}^A$ are the structure constants in the superconformal algebra and notice that we wrote flat indices everywhere. So, in order to calculate $\delta(\epsilon)\widehat{R}_{ab(L)}(Q)$, we need

$$\begin{aligned}
&[D, Q_\alpha] = \frac{1}{2}Q_\alpha, \qquad [U(1), Q_\alpha] = \frac{1}{2}\mathrm{i}(\gamma_5)_\alpha{}^\beta Q_\beta,\\
&[M_{ab}, Q_\alpha] = -\frac{1}{4}(\gamma_{ab})_\alpha{}^\beta Q_\beta, \qquad \delta\Psi_\mu^\beta = \ldots + \eta^\alpha(\gamma_\mu)_\alpha{}^\beta,\\
&\delta\phi_\mu^\beta = \ldots + \frac{1}{6}\mathrm{i}\epsilon^\alpha\left[(\gamma^\nu)_\alpha{}^\beta\tilde{\widehat{R}}_{\mu\nu}(\mathcal{A}) + (\gamma_5\gamma^\nu)_\alpha{}^\beta\widehat{R}_{\mu\nu}(\mathcal{A})\right].
\end{aligned} \quad (5.112)$$

The transformation of $\phi_\mu^\beta$ can be found after we impose the conventional constraints, which allow us to replace $f_\mu^a$ by an expression in terms of the independent gauge fields. If we combine (5.111) and (5.112), we find that

$$\begin{aligned}
\delta(\epsilon)\widehat{R}^\alpha_{ab(L)}(Q) &= \frac{1}{2}\widehat{R}_{ab}(D)\epsilon^\alpha_{(L)} + \frac{1}{2}\mathrm{i}\widehat{R}_{ab}(\mathcal{A})(\gamma_5)_\beta{}^\alpha\epsilon^\beta_{(L)} + \frac{1}{4}\widehat{R}_{ab}{}^{cd}(M)(\gamma_{cd})_\beta{}^\alpha\epsilon^\beta_{(L)}\\
&\quad -\frac{1}{3}\mathrm{i}\left[(\gamma^c\gamma_{[b})_\alpha{}^\beta\tilde{\widehat{R}}_{a]c}(\mathcal{A}) + (\gamma_5\gamma^c\gamma_{[b})_\alpha{}^\beta\widehat{R}_{a]c}(\mathcal{A})\right]\epsilon^\alpha_{(L)}\\
&= \epsilon^\beta_{(L)}\Big[\frac{1}{6}\mathrm{i}(\eta_{ac}\eta_{bd} + \frac{1}{2}\mathrm{i}\varepsilon_{abcd} - \gamma_{db}\eta_{ac} + \gamma_{da}\eta_{bc})_\beta{}^\alpha\widehat{R}^{cd}(\mathcal{A})\\
&\quad -\frac{1}{4}(\gamma^{cd})_\beta{}^\alpha\left(\widehat{R}_{abcd}(M) + 2\eta_{bd}\widehat{R}_{ac}(M) - 2\eta_{ad}\widehat{R}_{bc}(M)\right)\Big].
\end{aligned} \quad (5.113)$$

The second equality follows from the relations

$$\widehat{R}_{ab}(D) = -\widehat{R}_{acb}{}^{c}(M) = \frac{1}{6}\mathrm{i}\varepsilon_{abcd}\widehat{R}^{cd}(\mathcal{A})\,, \quad (5.114)$$

which are a consequence of the conventional constraints and Bianchi identities. The result in (5.113) can be simplified by using gamma-matrix identities. The final result is

$$\delta(\epsilon)\widehat{R}^{\alpha}_{ab(L)}(Q) = \epsilon^{\beta}_{(L)}\left[\frac{1}{2}\mathrm{i}(T_{cdab})_{\beta}{}^{\alpha}\widehat{R}^{cd}(\mathcal{A}) - \frac{1}{4}(\gamma^{cd})_{\beta}{}^{\alpha}W_{abcd}\right], \tag{5.115}$$

where $T_{abcd}$ is a projection operator, antisymmetric in the first two indices,

$$T_{abcd} = \frac{1}{6}\left(\eta_{ac}\eta_{bd} + \mathrm{i}\gamma_5\epsilon_{abcd} + \frac{1}{2}\gamma_{db}\eta_{ac} - \frac{1}{2}\gamma_{bc}\eta_{ad}\right) - (a \leftrightarrow b), \tag{5.116}$$

and $W_{abcd}$ is the usual expression for the Weyl tensor ($\widehat{R}_{ab}(M)$ is traceless):

$$W_{abcd} = \widehat{R}_{abcd}(M) + \eta_{bd}\widehat{R}_{ac}(M) - \eta_{ad}\widehat{R}_{bc}(M)\,. \tag{5.117}$$

Now we have determined the spinor component of the superconformal Weyl multiplet $\widehat{W}^{\alpha}_{ab}$:

$$\chi_{\beta(L)}(\widehat{W}^{\alpha}_{ab}) = \frac{1}{2}\mathrm{i}(T_{cdab})_{\beta}{}^{\alpha}R^{cd}(\mathcal{A}) - \frac{1}{4}(\gamma^{cd})_{\beta}{}^{\alpha}W_{abcd}\,. \tag{5.118}$$

In the same way, we can calculate $\delta(\epsilon)\chi_{(L)}(\widehat{W}^{\alpha}_{ab})$ to find $h(\widehat{W}^{\alpha}_{ab})$. It's easy to see that it should be proportional to $\widehat{R}_{ab}(S)$:

$$h(\widehat{W}^{\alpha}_{ab}) = -2T_{abcd}\widehat{R}^{cd}_{(L)}(S). \tag{5.119}$$

The next step is to calculate the components of $f(S)\widehat{W}_{ab\alpha}\widehat{W}^{ab}_{\beta}\varepsilon^{\alpha\beta}$. The multiplet calculus tells us that the product of any two chiral multiplets is again a chiral multiplet. More precisely, if we multiply the superconformal chiral multiplets $S_1$ and $S_2$ with components $(z_1, \chi_{1(L)}, h_1)$ and $(z_2, \chi_{2(L)}, h_2)$ respectively, we obtain a chiral multiplet $S_1 \cdot S_2$ with scalar component $z_1 z_2$. The other components can be determined via the supersymmetry transformations. One can show that

$$\chi_{(L)}(S_1 \cdot S_2) = \chi_{1(L)}z_2 + z_1\chi_{2(L)}\,, \tag{5.120}$$

$$h(S_1 \cdot S_2) = h_1 z_2 + z_1 h_2 - 2\bar{\chi}_{1(L)}\chi_{2(L)}\,. \tag{5.121}$$

More generally, if one considers an arbitrary function $g(S)$ of $m$ chiral multiplets $S^i = (z^i, \chi^i_{(L)}, h^i)$, $i = 1, \cdots, m$, then the components of $g(S)$ are

$$z(g(S)) = g(z)\,, \tag{5.122}$$

$$\chi_{(L)}(g(S)) = \partial_i g(z)\,\chi^i_{(L)}\,, \tag{5.123}$$

$$h(g(S)) = \partial_i g(z)\,h^i - \partial_{ij}g(z)\,\bar{\chi}^i\chi^j_{(L)}\,. \tag{5.124}$$

These rules will now be used to calculate the components of $f(S)\widehat{W}_{ab\alpha}\widehat{W}^{ab}_{\beta}\varepsilon^{\alpha\beta}$ where $f(S)$ is an arbitrary function of chiral multiplets $S^I = (X^I, \Omega^I_{(L)}, h^I)$, $I = 1, \cdots, n_C + 1$ (we include a compensating multiplet).

We start with $\widehat{W}^2 = \widehat{W}_{ab\alpha}\widehat{W}^{ab}_{\beta}\varepsilon^{\alpha\beta}$ and find

$$
\begin{aligned}
z(\widehat{W}^2) &= -\bar{\widehat{R}}_{ab}(Q)\widehat{R}^{ab}_{(L)}(Q)\,, && (5.125)\\
\chi_{(L)}(\widehat{W}^2) &= \left(\mathrm{i}\widehat{R}^{cd}(\mathcal{A})T_{cdab} - \frac{1}{2}\gamma^{cd}W_{abcd}\right)\widehat{R}^{ab}_{(L)}(Q)\,, && (5.126)\\
h(\widehat{W}^2) &= 4\bar{\widehat{R}}_{ab}(Q)\widehat{R}^{ab}_{(L)}(S) + \frac{1}{2}W_{abcd}W^{abcd} - \frac{1}{4}\mathrm{i}\varepsilon^{cdef}W_{abcd}W^{ab}{}_{ef}\\
&\quad + \frac{1}{6}\mathrm{i}\varepsilon_{abcd}R^{ab}(\mathcal{A})R^{cd}(\mathcal{A}) - \frac{1}{3}R_{ab}(\mathcal{A})R^{ab}(\mathcal{A}). && (5.127)
\end{aligned}
$$

Another straightforward application of (5.122) - (5.124) leads to

$$
\begin{aligned}
&f(S)\widehat{W}^2 = \qquad (5.128)\\
&\quad\Big\{-f(X)\bar{\widehat{R}}_{ab}(Q)\widehat{R}^{ab}_{(L)}(Q);\\
&\quad f(X)\left(\mathrm{i}\widehat{R}^{cd}(\mathcal{A})T_{cdab} - \frac{1}{2}\gamma^{cd}W_{abcd}\right)\widehat{R}^{ab}_{(L)}(Q) - \bar{\widehat{R}}_{ab}(Q)\widehat{R}^{ab}_{(L)}(Q)\partial_I f\,\Omega^I_{(L)}\,;\\
&\quad -\left(\partial_I f\,h^I - \partial_{IJ}f\,\bar{\Omega}^I\Omega^J_{(L)}\right)\bar{\widehat{R}}_{ab}(Q)\widehat{R}^{ab}_{(L)}(Q) + f(X)\Big(4\bar{\widehat{R}}_{ab}(Q)\widehat{R}^{ab}_{(L)}(S)\\
&\quad + \frac{1}{2}W_{abcd}W^{abcd} - \frac{1}{4}\mathrm{i}\varepsilon^{cdef}W_{abcd}W^{ab}{}_{ef} + \frac{1}{6}\mathrm{i}\varepsilon_{abcd}R^{ab}(\mathcal{A})R^{cd}(\mathcal{A})\\
&\quad -\frac{1}{3}R_{ab}(\mathcal{A})R^{ab}(\mathcal{A})\Big) - 2\partial_I f\,\bar{\Omega}^I_{(L)}\left(\mathrm{i}\widehat{R}^{cd}(\mathcal{A})T_{cdab} - \frac{1}{2}\gamma^{cd}W_{abcd}\right)\widehat{R}^{ab}_{(L)}(Q)\Big\}\,.
\end{aligned}
$$

This is the result we were aiming for, and the components of $f\widehat{W}^2$ can now be substituted into the density formula (5.89) in order to obtain the superconformal invariant action. Since the final result is quite lengthy and we do not need the full expression, we will only write down the relevant terms:

$$\begin{aligned}\widehat{S}_{\text{Weyl}} &= \int \mathrm{d}^4x\,\frac{1}{2}ef(X)\left[4\bar{\widehat{R}}_{\mu\nu}(Q)\widehat{R}^{\mu\nu}_{(L)}(S)+\frac{1}{2}W_{\mu\nu ab}W^{\mu\nu ab}\right.\\ &\qquad -\frac{1}{4}\mathrm{i}e^{-1}\varepsilon^{\mu\nu\rho\sigma}W_{\mu\nu ab}W_{\rho\sigma}{}^{ab}+\frac{1}{6}\mathrm{i}e^{-1}\varepsilon^{\mu\nu\rho\sigma}R_{\mu\nu}(\mathcal{A})R_{\rho\sigma}(\mathcal{A})\\ &\qquad -\frac{1}{3}R_{\mu\nu}(\mathcal{A})R^{\mu\nu}(\mathcal{A})-\frac{1}{2}\bar{\Psi}_{\mu(R)}\gamma^{\mu\nu}\Psi_{\nu(R)}\bar{\widehat{R}}_{\rho\sigma}(Q)\widehat{R}^{\rho\sigma}_{(L)}(Q)\\ &\qquad \left.+\bar{\Psi}_{\mu(R)}\gamma^{\mu}\left(\mathrm{i}\widehat{R}^{\nu\rho}(\mathcal{A})T_{\nu\rho\lambda\sigma}-\frac{1}{2}\gamma^{ab}W_{\lambda\sigma ab}\right)\widehat{R}^{\lambda\sigma}_{(L)}(Q)\right]+\dots\\ &= \int \mathrm{d}^4x\ \mathrm{Im}\,f(X)\varepsilon^{\mu\nu\rho\sigma}\left[-\frac{1}{6}R_{\mu\nu}(\mathcal{A})R_{\rho\sigma}(\mathcal{A})+\frac{1}{4}R_{\mu\nu ab}R_{\rho\sigma}{}^{ab}\right.\\ &\qquad \left.+2\partial_\mu\left(2\bar{\phi}_\nu\mathcal{D}_\rho\Psi_\sigma-\bar{\phi}_\nu\gamma_\rho\phi_\sigma\right)\right]+\dots \end{aligned} \tag{5.129}$$

where $R_{\mu\nu}(\mathcal{A})=2\partial_{[\mu}\mathcal{A}_{\nu]}$ and $\mathcal{D}_\rho\Psi_\sigma$ is covariant with respect to all linearly realized symmetries:

$$\mathcal{D}_{[\mu}\Psi_{\nu]}=\left(\partial_{[\mu}+\frac{1}{2}b_{[\mu}+\frac{1}{4}\omega_{[\mu}{}^{ab}\gamma_{ab}+\frac{1}{2}\mathrm{i}\gamma_5\mathcal{A}_{[\mu}\right)\Psi_{\nu]}\,. \tag{5.130}$$

We note that the quantity between the square brackets in the last two lines of (5.129) is a total derivative and a superconformal invariant. All the other terms (schematically denoted by the dots in the last line of (5.129)) are either proportional to the real part of $f(X)$ or they depend on scalar derivatives of this function.

The final step in the superconformal program is the reduction of $\widehat{S}_{\text{Weyl}}$ to an ordinary Poincaré supergravity. This requires the gauge fixing of those symmetries that are not in the super-Poincaré group and the elimination of the auxiliary fields. As in ordinary superconformal calculus (without higher derivatives), the special conformal symmetry can be fixed by choosing $b_\mu=0$. This eliminates $b_\mu$ from $\mathcal{D}_\mu\Psi_\nu$. To fix dilatations, chiral $U(1)$ gauge transformations and special supersymmetry, one singles out a scalar multiplet (the so-called compensating scalar multiplet) whose components are then fixed, except for its highest component which becomes the auxiliary field in Poincaré supergravity. Since $f(X)$ has Weyl and chiral weight zero, it does not depend on the conformon scalar which carries the Weyl and chiral weight of the $X^I$. Therefore, after gauge fixing, $f$ becomes a function of the physical scalars $z^i$, $i=1,\dots,n_C$ only, i.e., $f=f(z)$. Finally, we should still mention one issue that makes the gauge fixing procedure in the presence of higher derivative corrections slightly more complicated. The field $\mathcal{A}_\mu$, which is an auxiliary field in ordinary superconformal calculus, now appears with higher derivative terms in the action (5.129). In particular, the superconformal action $\widehat{S}_{\mathrm{Weyl}}$ contains a kinetic term for $\mathcal{A}_\mu$:

$$\widehat{S}_{\mathrm{kin},\,\mathcal{A}} = -\frac{1}{3}\int \mathrm{d}^4x\, e\,[\mathrm{Re} f(X)]\, R_{\mu\nu}(\mathcal{A})R^{\mu\nu}(\mathcal{A})\,, \tag{5.131}$$

and one would expect that the field is now propagating. This problem of extra propagating degrees of freedom has already been discussed briefly below equation (5.19), and an adequate solution requires the addition of extra terms $\int \mathrm{d}^4\theta\left(\frac{1}{2}E_{\alpha\dot{\alpha}}E^{\alpha\dot{\alpha}} - 2RR^*\right)$ to the original Lagrangian. Indeed, it has been shown in [96] that these extra contributions exactly cancel the kinetic term $\widehat{S}_{\mathrm{kin},\,\mathcal{A}}$, i.e.,

$$\int \mathrm{d}^4\theta\left(\frac{1}{2}E_{\alpha\dot{\alpha}}E^{\alpha\dot{\alpha}} - 2RR^*\right) = \frac{1}{3}\int \mathrm{d}^4x\, e\,[\mathrm{Re} f(X)]\, R_{\mu\nu}(\mathcal{A})R^{\mu\nu}(\mathcal{A}) + \ldots\,. \tag{5.132}$$

For now we will keep writing $\mathcal{A}_\mu$, although it should be seen as a composite field.

Finally, if we work through the entire gauge fixing procedure, we find the following relevant terms in the Poincaré invariant action (which will be denoted without a hat, i.e., $S_{\mathrm{Weyl}}$):

$$\begin{aligned} S_{\mathrm{Weyl}} \;=\; & \int \mathrm{d}^4x\ \mathrm{Im}\, f(z)\varepsilon^{\mu\nu\rho\sigma}\left[-\frac{1}{6}R_{\mu\nu}(\mathcal{A})R_{\rho\sigma}(\mathcal{A}) + \frac{1}{4}R_{\mu\nu ab}R_{\rho\sigma}{}^{ab}\right. \\ & \left. + 2\partial_\mu\left(2\bar{\phi}_\nu\mathcal{D}_\rho\Psi_\sigma - \bar{\phi}_\nu\gamma_\rho\phi_\sigma\right)\right] + \ldots\,, \end{aligned} \tag{5.133}$$

with

$$\begin{aligned} \mathcal{D}_{[\mu}\Psi_{\nu]} &= \left(\partial_{[\mu} + \frac{1}{4}\omega_{[\mu}{}^{ab}(e)\gamma_{ab} + \frac{1}{2}\mathrm{i}\gamma_5\mathcal{A}_{[\mu}\right)\Psi_{\nu]}\,, \\ \phi_\mu &= \frac{1}{2}R_\mu - \frac{1}{6}\gamma_\mu\gamma\cdot R\,, \\ R^\mu &= e^{-1}\varepsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu\mathcal{D}_\rho\Psi_\sigma\,. \end{aligned} \tag{5.134}$$

In the first line of (5.133) we recognize the Hirezbruck signature density, which is the counterterm that led to a cancellation of the gravitational anomaly in the simple example (i.e. $f(z) = z$) of §5.2. In addition, there are extra terms that couple to the imaginary part of the holomorphic function $f$, namely the first term in (5.133) which is a Peccei-Quinn-like term for $\mathcal{A}_\mu$ (but with the opposite sign), and terms that contain two derivatives on the gravitino. All these extra terms are necessarily involved in a Green-Schwarz cancellation mechanism that has its origin in a non-trivial variation of the function $\mathrm{Im}\, f$ under gauge transformations of the scalar fields. The terms that were omitted in (5.133) depend on the real part of the scalar function $f$, or on its scalar derivatives.

### The cancellation

Any Green-Schwarz mechanism involves a non-vanishing gauge transformation of the classical action on the one side, and an anomalous transformation of the quantum effective action on the other side. In the case at hand, the anomaly has been studied in §5.7 and we found that only Abelian factors in the gauge group play a role. In other words, only Abelian transformations of the effective action are non-vanishing (in the absence of pure gauge anomalies). We also know that the anomaly is proportional to the signature density $\varepsilon^{\mu\nu\rho\sigma} R_{\mu\nu ab} R_{\rho\sigma}{}^{ab}$.

The other ingredient, the gauge transformation of the action $S_{\text{Weyl}}$, has not been studied so far. First, we note that the only source for a possible gauge variance comes from the coupling to the scalar multiplets. All the other fields sit in the supergravity multiplet and are supposed to be uncharged under the gauge group. Moreover, due to the particular form of the anomaly, we have to meet the following requirements:

- The action $S_{\text{Weyl}}$ should only transform under Abelian gauge transformations.
- We expect that only $\operatorname{Im} f$ transforms, and that both $\operatorname{Re} f$ and scalar derivatives of $f$ are invariant.
- The resulting gauge transformation of $\operatorname{Im} f$ should be independent of the scalars.

These conditions put strong constraints on the possible holomorphic functions $f(z)$. The easiest case where all three requirements are satisfied is when $f(z)$ depends arbitrarily on the scalars that are uncharged and linearly on those that transform with an imaginary shift. Schematically we write

$$f(z) = g(t) + a_j s^j(x)\,, \tag{5.135}$$

where $g(t)$ is an arbitrary function of the neutral scalars $t^k$, the parameters $a_j$ are real constants and $s^j(x)$ are the scalar fields that transform with an imaginary shift under the different $U(1)$ factors in the gauge group:

$$\delta(\Lambda_{(s)}) s^j(x) = \mathrm{i} m^j_{(s)}\, \Lambda_{(s)}(x)\,. \tag{5.136}$$

Here $\Lambda_{(s)}(x)$ is a local parameter that corresponds to the $U(1)_s$ gauge group and the $m^j_{(s)}$ are real constants (not necessarily different from zero). In fact this situation is nothing but a generalization of our example in §5.2. There we considered only one scalar field $s^1 = z$ with $f(z) = z$ and a transformation $\delta(\Lambda) z = \mathrm{i} m \Lambda(x)$ under the Abelian gauge group (recall (5.22)). The more general form for $f(z)$ in equation (5.135) also follows from generic string theory compactifications, which is an additional check on its validity.

If we now substitute $f(z)$ in the action and calculate its gauge transformation, we find that

$$\begin{aligned}\delta(\Lambda)S_{\text{Weyl}} &= \sum_s a_j m^j_{(s)} \int \mathrm{d}^4x\, \Lambda_{(s)}\varepsilon^{\mu\nu\rho\sigma}\Big[-\frac{1}{6}R_{\mu\nu}(\mathcal{A})R_{\rho\sigma}(\mathcal{A}) + \frac{1}{4}R_{\mu\nu ab}R_{\rho\sigma}{}^{ab} \\ &\qquad + 2\partial_\mu\left(2\bar{\phi}_\nu\mathcal{D}_\rho\Psi_\sigma - \bar{\phi}_\nu\gamma_\rho\phi_\sigma\right)\Big]\,. \qquad (5.137)\end{aligned}$$

The second term in this variation looks exactly like the mixed gauge-gravitational anomaly in (5.104) and cancellation occurs, provided that the following constraint is satisfied:

$$-48\pi^2 a_j m^j_{(s)} = \sum_\ell q^{(s)}_\ell\,. \qquad (5.138)$$

This is a generalization of (5.33) to multiple scalars and multiple $U(1)$-factors. However, the cancellation is not complete because the first and third term in (5.137) do not have an obvious equivalent on the anomaly-side. The contribution that is proportional to $\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\mathcal{A}_\nu\partial_\rho\mathcal{A}_\sigma$ might be related to the cancellation of a mixed $U(1)_s$-Kähler anomaly, but the meaning of the fermionic terms in (5.137) is certainly not clear at the moment.

## 5.8 Conclusions and outlook

In this chapter we have studied the consistency conditions that ensure the gauge and supersymmetry invariance of matter coupled $\mathcal{N} = 1$ supergravity theories. In the absence of mixed gauge-gravitational anomalies, this requires the interplay between Peccei-Quinn terms, generalized Chern-Simons terms and pure gauge anomalies. Each of these three ingredients defines a constant three-index tensor:

1. The gauge non-invariance of the Peccei-Quinn terms is proportional to a constant imaginary shift of the gauge kinetic function parameterized by a tensor $X_{\Omega\Lambda\Sigma}$. This tensor in general splits into a completely symmetric part and a part of mixed symmetry, $X^{(\mathrm{s})}_{\Omega\Lambda\Sigma} + X^{(\mathrm{m})}_{\Omega\Lambda\Sigma}$.

2. Generalized Chern-Simons terms are defined by a tensor, $X^{(\mathrm{CS})}_{\Omega\Lambda\Sigma}$, of mixed symmetry.

3. Quantum gauge anomalies of chiral fermions are proportional to a tensor $d_{\Lambda\Sigma\Omega}$, which, in the appropriate regularization scheme, can be chosen to be completely symmetric.

We find the full quantum effective action to be gauge invariant if

$$X_{\Omega\Lambda\Sigma} = X^{(\mathrm{CS})}_{\Omega\Lambda\Sigma} + d_{\Lambda\Sigma\Omega}\,. \tag{5.139}$$

The compatibility of this result with supersymmetry is non-trivial, because a violation of gauge symmetries usually also triggers a violation of the on-shell supersymmetry, as is best seen by recalling that in the Wess-Zumino gauge the preserved supersymmetry is a combination of the original superspace supersymmetry and a gauge transformation. Due to the presence of the quantum gauge anomalies, one therefore also has to take into account the corresponding supersymmetry anomalies of the quantum effective action. However, we could show that supersymmetry invariance of the full quantum effective action requires exactly the same condition (5.139).

Besides the cancellation of gauge and supersymmetry anomalies in both global and local supersymmetry, we have also studied an extension of the Green-Schwarz mechanism to $\mathcal{N} = 1$ supergravities with a non-trivial mixed anomaly. The counterterms, which are higher derivative terms that are not present in the standard action, depend linearly on the scalars and cancel the anomaly if the condition in equation (5.138) is satisfied. However, our supersymmetric treatment also reveals left-over transformations that do not obviously cancel any anomaly. This unsolved issue requires an extra investigation in the future.

Our results are interesting for a number of rather different applications. In [10], orientifold compactifications with anomalous fermion spectra were studied, in which the chiral anomalies are canceled by a mixture of the Peccei-Quinn and generalized Chern-Simons terms. The analysis in [10] was mainly concerned with the gauge invariance of the bosonic part of the action and revealed the generic presence of a completely symmetric and a mixed part in $X_{\Omega\Lambda\Sigma}$ and the generic necessity of generalized Chern-Simons terms. Our results show how such theories can be embedded into the framework of $\mathcal{N} = 1$ supergravity and supplements the phenomenological discussions of [10] by the fermionic couplings in a supersymmetric setting.

The work of [10] also raises the general question of the possible higher-dimensional origins of GCS terms. In [12], certain flux and generalized Scherk-Schwarz compactifications [118,119] are identified as another means to generate such terms. In [120], it was also shown that $\mathcal{N} = 2$ supergravity theories with GCS terms can be obtained by ordinary dimensional reduction of certain $5D$, $\mathcal{N} = 2$ supergravity theories with tensor multiplets [43, 121]. It would be interesting to obtain a more complete picture of the possible origins of GCS-terms in string theory and supergravity theories.

Finally, in reference [61], a general set-up for treating gauged supergravities in a manifestly symplectic framework was proposed. In this work, which we have reviewed in §4.3, the completely symmetric part of what we call $X_{\Omega\Lambda\Sigma}$ was assumed to be zero, following the guideline of extended supergravity theories. As we emphasized in this chapter, $\mathcal{N} = 1$ supergravity theories might allow for a non-vanishing $X^{(s)}_{\Omega\Lambda\Sigma}$, and hence a possible extension of the set-up of [61] and §4.3 in the presence of quantum anomalies. Such an extension does indeed exist, and it will be the content of the next chapter.

CHAPTER 6

# ANOMALY CANCELLATION AND GENERALIZED GAUGINGS

*We present a generalization of the results from chapter 5 to theories that are manifestly electric/magnetic duality covariant. Equivalently, this can be seen as an extension of the embedding tensor formalism in §4.3 to theories with quantum anomalies. This work has been carried out in collaboration with Torsten T. Schmidt, Mario Trigiante, Antoine Van Proeyen and Marco Zagermann and was published in [2].*

## 6.1 Introduction

In the previous chapter it was shown how (i) anomalous fermionic spectra, (ii) Peccei-Quinn terms with gauged axionic shift symmetries, and (iii) generalized Chern-Simons (GCS) terms can be compatible with global and local $\mathcal{N} = 1$ supersymmetry. While we discussed the general interplay between all three ingredients, it should be emphasized that not all three ingredients necessarily need to be present in a gauge invariant theory. Indeed, one can construct purely classical theories, in which only the last two ingredients (ii) and (iii), i.e. the gauged shift symmetries and the GCS terms, are present and the fermionic spectrum is either absent or non-anomalous. This corresponds to a vanishing anomaly tensor $d_{\Lambda\Sigma\Omega}$ in the previous chapter, and the constraint (5.139) that ensures gauge (and supersymmetry) invariance of the effective action reduce to

$$X^{(\mathrm{m})}_{\Omega\Lambda\Sigma} = X^{(\mathrm{CS})}_{\Omega\Lambda\Sigma}\,, \qquad\qquad X^{(\mathrm{s})}_{\Omega\Lambda\Sigma} = 0\,. \tag{6.1}$$

In fact, it was in such a context that GCS terms were first discussed in the literature. More concretely, their possibility was first discovered in extended gauged supergravity theories [41], which are automatically free of quantum anomalies due to the incompatibility of chiral gauge interactions with extended $4D$ supersymmetry.

Another very important example in this context is the work of H. Nicolai, H. Samtleben, M. Trigiante and B. de Wit on theories with generalized gaugings. Recall from §4.3 that such theories combine classically gauge invariant local Lagrangians – that may also include Peccei-Quinn and GCS terms – with the concept of electric/magnetic duality transformations. The presence of a GCS term in these theories is not surprising because their gauge kinetic function transforms with a non-trivial shift, i.e., the term proportional to $X_{M\Lambda\Sigma}$ in (4.83). The electric component of $X_{M\Lambda\Sigma}$ is given by $X_{\Omega\Lambda\Sigma}$ and can be identified with the shift tensor in theories with an electrical gauging. Therefore, the constraints in (6.1) guarantee a non-trivial coefficient for the GCS terms, and a vanishing totally symmetric combination $X_{(\Omega\Lambda\Sigma)}$. A generalization of (6.1) to include also the magnetic component of the shift tensor $X_{M\Lambda\Sigma}$ was implemented in §4.3 as follows:

- ▸ The Chern-Simons action in (4.103) is written in terms of the symplectic matrices $X_M$. In this way, the identification $X^{(\mathrm{m})}_{\Omega\Lambda\Sigma} = X^{(\mathrm{CS})}_{\Lambda\Sigma\Omega}$ and its magnetic generalization were immediately implemented.

- ▸ A generalization of the second condition in (6.1) was imposed via the representation constraint, recall (4.105):

  $$D_{MNP} \equiv X_{(MN}{}^{Q}\Omega_{P)Q} = 0\,. \tag{6.2}$$

  Indeed, if one writes (6.2) into its electric and magnetic components, one of the components is precisely given by $X^{(\mathrm{s})}_{\Omega\Lambda\Sigma} = 0$.

Together with the closure constraint, (4.85), the representation constraint (6.2) is necessary and sufficient to show gauge invariance of the extended action $S_{\mathrm{VT}} = S_{\mathrm{g.k.}} + S_{\mathrm{GCS}} + S_{\mathrm{top},\,B}$ (recall (4.101)) under general transformations of the electric/magnetic vector fields, 2-forms and the gauge kinetic function.

However, the full physical meaning of (6.2) always remained a bit obscure, and was inferred in the literature [65, 73, 83] from identities that are known to be valid in $\mathcal{N} = 8$ or $\mathcal{N} = 2$ supergravity. In this chapter we propose a physical interpretation of the representation constraint as we will we recognize (6.2) as the condition for the *absence of quantum anomalies*. Since quantum anomalies are automatically absent in extended $4D$ supergravity theories, it is no surprise that the internal consistency of $\mathcal{N}=8$ or $\mathcal{N}=2$ supergravity always hinted at the validity of (6.2).

We then go one step further and show that if quantum anomalies proportional to a constant, totally symmetric tensor,[1] $d_{MNP}$, are present, the representation constraint (6.2) has to be relaxed to

$$X_{(MN}{}^{Q}\Omega_{P)Q} = d_{MNP}\,, \qquad \text{with} \qquad d_{MNP} = \Theta_M{}^{\alpha}\Theta_N{}^{\beta}\Theta_P{}^{\gamma}d_{\alpha\beta\gamma}\,, \tag{6.3}$$

to allow for a gauge invariant quantum effective action. Here $d_{\alpha\beta\gamma}$ is a symmetric tensor that will be defined by the anomalies. We show explicitly how the framework in §4.3 has to be modified in such a situation and the resulting gauge variance of the classical Lagrangian precisely gives the negative of the consistent quantum anomaly encoded in $d_{MNP}$.

To summarize, our work can be viewed as a generalization of §4.3 to theories with quantum anomalies. Equivalently, it is the covariantization of our results in chapter 5 with respect to electric/magnetic duality transformations. Our newly proposed constraint (6.3) generalizes the anomaly cancellation condition in (5.74), and includes situations in which pseudo-anomalous gauge interactions are mediated by magnetic vector potentials. While already interesting in itself, our results promise to be very useful for the description of flux compactifications with chiral fermionic spectra, as e.g. in intersecting brane models on orientifolds with fluxes, because flux compactifications often give $4D$ theories which appear naturally in unusual duality frames and contain two-form fields.

The outline of this chapter is as follows. In §6.2, we present the different steps that show gauge invariance of the covariant action $S_{\rm VT} = S_{\rm g.k.} + S_{\rm GCS} + S_{{\rm top},\,B}$ in the absence of gauge anomalies (this is the framework as we presented it in §4.3, including the original representation constraint $D_{MNP}=0$). Next, in §6.3 we show how the formalism of §4.3 has to be modified in order to accommodate quantum anomalies involving the relaxed representation constraint (6.2). In particular, we reconsider the gauge transformation of the action $S_{\rm VT}$ and show how it leads to a non-trivial result that is precisely the negative of the quantum anomaly. We flesh out our results with a simple non-trivial example in §6.4 and conclude in §6.5.

## 6.2 Gauge invariance of $S_{\rm VT}$

The action for vectors and 2-forms in the electric/magnetic covariant formalism was presented in (4.101). It contains three parts; the kinetic action $S_{\rm g.k.}$, the

[1] The tensor $d_{MNP}$ is the one that defines the consistent anomaly in the form given in equation (6.25). As the gauge symmetry in the matter sector is implemented by minimal couplings to the gauge potentials dressed with an embedding tensor, as can be seen from (4.80), the tensor $d_{MNP}$ must be of the form (6.3).

generalized Chern-Simons terms $S_{\rm GCS}$, and the topological terms $S_{{\rm top},\,B}$. We will calculate the gauge variation of each of these contributions and show that the total variation vanishes, given the closure and representation constraints, $Q_{MN}{}^{\alpha}$ and $D_{MNP}=0$ respectively.

### The kinetic action

The kinetic Lagrangian $\mathcal{L}_{\rm g.k.}$ can be written as

$$\mathcal{L}_{\rm g.k.} = -\frac{1}{8}\varepsilon^{\mu\nu\rho\sigma}\mathcal{H}^{\Lambda}_{\mu\nu}\mathcal{G}_{\rho\sigma\,\Lambda}\,, \tag{6.4}$$

where we used the definition (4.108) for the dual field strength $\mathcal{G}_{\mu\nu\Lambda}$. The variation of $\mathcal{L}_{\rm g.k.}$ can be computed using the infinitesimal transformation of the gauge kinetic function $\mathcal{N}_{\Lambda\Sigma}$ (see (4.83)). We find that

$$\begin{aligned}\delta\mathcal{L}_{\rm g.k.} = &\\ &-\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\mathcal{G}_{\mu\nu\,\Lambda}\delta\mathcal{H}^{\Lambda}_{\rho\sigma}\\ &+\frac{1}{8}\varepsilon^{\mu\nu\rho\sigma}\Lambda^{Q}\left(\mathcal{H}^{\Lambda}_{\mu\nu}X_{Q\Lambda\Sigma}\mathcal{H}^{\Sigma}_{\rho\sigma}-2\mathcal{H}^{\Lambda}_{\mu\nu}X_{Q\Lambda}{}^{\Sigma}\mathcal{G}_{\rho\sigma\,\Sigma}-\mathcal{G}_{\mu\nu\,\Lambda}X_{Q}{}^{\Lambda\Sigma}\mathcal{G}_{\rho\sigma\,\Sigma}\right)\,.\end{aligned} \tag{6.5}$$

The first line of this result can be further worked out if we first compute the variation of the field strengths $\mathcal{H}_{\mu\nu}{}^{M}$ under general transformations of the vectors and two-forms. In equation (4.98) it was shown that $\mathcal{H}_{\mu\nu}{}^{M}$ transforms covariantly if $\Delta B_{\mu\nu}{}^{MN}$ in (4.106) vanishes. However, if $\Delta B_{\mu\nu}{}^{MN}\neq 0$ (which will generally be the case), this covariant transformations needs to be modified to

$$\delta\mathcal{H}_{\mu\nu}{}^{M} = X_{NQ}{}^{M}\Lambda^{Q}\mathcal{H}_{\mu\nu}{}^{N} + Y^{M}{}_{NP}\Delta B_{\mu\nu}{}^{NP}\,. \tag{6.6}$$

If we plug this variation into (6.5), and also use the definition $\mathcal{G}_{\mu\nu}{}^{M} = (\mathcal{H}_{\mu\nu}{}^{\Lambda},\mathcal{G}_{\mu\nu\Lambda})$, we find

$$\begin{aligned}\delta\mathcal{L}_{\rm g.k.} &= \varepsilon^{\mu\nu\rho\sigma}\ \Big[-\frac{1}{4}\mathcal{G}_{\mu\nu\,\Lambda}\left(\Lambda^{Q}X_{PQ}{}^{\Lambda}\mathcal{H}_{\rho\sigma}{}^{P}+Y^{\Lambda}{}_{NP}\Delta B_{\rho\sigma}{}^{NP}\right)\\ &\qquad\qquad + \frac{1}{8}\mathcal{G}_{\mu\nu}{}^{M}\mathcal{G}_{\rho\sigma}{}^{N}\Lambda^{Q}X_{QM}{}^{R}\Omega_{NR}\Big]\,.\end{aligned} \tag{6.7}$$

Note that the second line is now a covariant expression.

Clearly, the proposed form for the kinetic action is not gauge invariant. This should not come as a surprise because $\delta\mathcal{N}_{\Lambda\Sigma}$ contains a constant shift, which requires the addition of generalized Chern-Simons terms to the action, as was reviewed in §5.3 for purely electric gaugings. Also the last term on the right hand side of (4.83) gives extra contributions that are quadratic in the kinetic function. In the next steps we will see that besides the GCS terms, also topological terms are necessary to cancel the non-vanishing result in (6.7). Let us first study the gauge transformation of the GCS terms.

### Generalized Chern-Simons terms

Modulo total derivatives, the variation of $\mathcal{L}_{\rm GCS}$ can be written as

$$\begin{aligned}\delta\mathcal{L}_{\rm GCS} \quad &= \varepsilon^{\mu\nu\rho\sigma} \quad \Big[\frac{1}{2}\mathcal{F}_{\mu\nu}{}^{\Lambda} D_\rho \delta A_{\sigma\Lambda} - \frac{1}{2}\mathcal{F}_{\mu\nu\Lambda} Y^{\Lambda}{}_{NP} A_\rho{}^N \delta A_\sigma{}^P \\ &\qquad -D_{MNP} A_\mu{}^M \delta A_\nu{}^N \left(\partial_\rho A_\sigma{}^P + \frac{3}{8} X_{RS}{}^P A_\rho{}^R A_\sigma{}^S\right)\Big]\,. \end{aligned} \tag{6.8}$$

We used (4.88) antisymmetrized in $[MNQ]$ and the definition of $D_{MNP}$. In fact, the last line of (6.8) vanishes if we impose the linear constraint $D_{MNP} = 0$. For later convenience, though, we will gather all terms that are proportional to $D_{MNP}$.

### Topological terms for the $B$-field

The last step towards gauge invariance is made by adding topological terms linear and quadratic in the tensor field $B_{\mu\nu}{}^{NP}$ to the gauge kinetic term and generalized Chern-Simons action.

The variation of $\mathcal{L}_{\rm top,\,B}$ is

$$\begin{aligned}\delta\mathcal{L}_{\rm top,\,B} \quad &= \quad \frac{1}{4}\,\varepsilon^{\mu\nu\rho\sigma} Y^{\Lambda}{}_{NP}\left[\mathcal{H}_{\mu\nu\Lambda}\,\delta B_{\rho\sigma}{}^{NP} + B_{\rho\sigma}{}^{NP}\delta\mathcal{F}_{\mu\nu\Lambda}\right] \\ &= \quad \frac{1}{4}\,\varepsilon^{\mu\nu\rho\sigma} Y^{\Lambda}{}_{NP}\Big[\mathcal{H}_{\mu\nu\Lambda}\,\delta B_{\rho\sigma}{}^{NP} \\ &\qquad + 2B_{\rho\sigma}{}^{NP}\left(D_\mu \delta A_{\nu\Lambda} - Y_{\Lambda RS} A_\mu^R \delta A_\nu^S\right)\Big]\,. \end{aligned} \tag{6.9}$$

This expression can be combined with (6.8) to

$$\begin{aligned}&\delta\left(\mathcal{L}_{\rm top,\,B} + \mathcal{L}_{\rm GCS}\right) = \\ &\quad \varepsilon^{\mu\nu\rho\sigma}\Big[\frac{1}{2}\mathcal{H}_{\mu\nu}{}^{\Lambda} D_\rho \delta A_{\sigma\Lambda} + \frac{1}{4}\mathcal{H}_{\mu\nu\Lambda} X_{(NP)}{}^{\Lambda}\left(\delta B_{\rho\sigma}{}^{NP} - 2A_\rho{}^N \delta A_\sigma{}^P\right) \\ &\qquad -D_{MNP} A_\mu{}^M \delta A_\nu{}^N \left(\partial_\rho A_\sigma{}^P + \frac{3}{8} X_{RS}{}^P A_\rho{}^R A_\sigma{}^S\right)\Big]\,. \end{aligned} \tag{6.10}$$

### Variation of the total action

We are now ready to discuss the symmetry variation of the total Lagrangian

$$\mathcal{L}_{\mathrm{VT}} = \mathcal{L}_{\mathrm{g.k.}} + \mathcal{L}_{\mathrm{top},B} + \mathcal{L}_{\mathrm{GCS}}\,. \tag{6.11}$$

But before we combine the results from sections 6.2 - 6.2 to obtain $\delta\mathcal{L}_{\mathrm{VT}}$, it is beneficial to introduce some useful relations that simplify the calculations. First we use the definition (6.2) to check that

$$2Y^Q{}_{MN}\Omega_{RQ} + X_{RM}{}^Q\Omega_{NQ} = 3D_{MNR}\,,$$

$$\text{i.e.}\qquad Y^P{}_{MN} = \frac{1}{2}\Omega^{PR}X_{RM}{}^Q\Omega_{NQ} + \frac{3}{2}D_{MNR}\Omega^{RP}\,. \tag{6.12}$$

This relation leads to the following important equalities:

$$Y^M{}_{NP}\Omega_{MQ}Y^Q{}_{RS} = -\frac{3}{2}D_{NPQ}Y^Q{}_{RS} + \frac{1}{2}Q_{(RS)}{}^\alpha\Delta_{\alpha NP}\,, \tag{6.13}$$

or

$$\begin{aligned} Y^M{}_{NP}\Omega_{MQ}Y^Q{}_{RS} &= \frac{1}{4}E^{\alpha\beta}\Delta_{\alpha NP}\Delta_{\beta RS} + \frac{9}{4}D_{NPM}\Omega^{MQ}D_{RSQ} \\ &\quad + \frac{3}{2}Z^{M\alpha}\left(\Delta_{\alpha NP}D_{RSM} - \Delta_{\alpha RS}D_{NPM}\right)\,. \end{aligned} \tag{6.14}$$

Here, we introduced two objects that will frequently be used in the future, namely

$$Z^{M\alpha} \equiv \frac{1}{2}\Omega^{MN}\Theta_N{}^\alpha\,, \qquad \text{and} \qquad \Delta_{\alpha NP} \equiv (t_\alpha)_{(N}{}^M\Omega_{P)M}\,. \tag{6.15}$$

From (6.13) and (6.14) we see that $Y^M{}_{NP}\Omega_{MQ}Y^Q{}_{RS}$ can be written in two equivalent ways. In (6.13) the right hand side is linear in the representation and closure constraints, whereas the right hand side of (6.14) contains the representation and locality constraints. If the constraints are satisfied, then $Y^M{}_{NP}\Omega_{MQ}Y^Q{}_{RS}$ vanishes in both cases. As a bonus, we find a relationship between $D_{MNP}$, $Q_{(RS)}{}^\alpha$ and $E^{\alpha\beta}$ upon elimination of $Y^M{}_{NP}\Omega_{MQ}Y^Q{}_{RS}$. This approves point 4 in §4.3, where we remarked that the constraints are not all independent.

In the remainder of this section we will use (6.13) and only impose the closure constraint $Q_{(RS)}{}^\alpha = 0$ such that

$$Y^M{}_{NP}\Omega_{MQ}Y^Q{}_{RS} = -\frac{3}{2}D_{NPQ}Y^Q{}_{RS}\,. \tag{6.16}$$

For the time being we will not impose the representation constraint, but instead we gather all terms that are proportional to $D_{MNP}$. This will be useful once we make the modification (6.3) later on in this chapter. Only in the final result for $\delta\mathcal{L}_{\mathrm{VT}}$ we will set $D_{MNP}=0$.

Let us now calculate the complete variation of (6.11). We start with the $\Xi$-transformations. We see immediately from (6.7) that the gauge-kinetic terms are invariant. Using (4.93), the second line of (6.10) gives a non-vanishing contribution that is proportional to $D_{MNP}$. Together with the first line of (6.10), which, using (4.106) and (4.93), can be written in a symplectically covariant form, we obtain:

$$\begin{aligned}\delta(\Xi)\mathcal{L}_{\mathrm{VT}} \;=\;& \varepsilon^{\mu\nu\rho\sigma}\Big[D_{MNP}Y^N{}_{TU}\Xi_\nu{}^{TU}A_\mu{}^M\Big(\partial_\rho A_\sigma{}^P+\frac{3}{8}X_{RS}{}^PA_\rho{}^RA_\sigma{}^S\Big)\\ &-\frac{1}{2}\mathcal{H}_{\mu\nu}{}^MY^Q{}_{NP}\Omega_{MQ}D_\rho\Xi_\sigma{}^{NP}\Big]\,. \end{aligned}\tag{6.17}$$

In the second line of (6.17), the $B$-terms in $\mathcal{H}$ are proportional to $Y^M{}_{RS}$ (see (4.97)) and thus give a contribution that is proportional to $D_{MNP}$, due to our new relation (6.16). For the $\mathcal{F}$-terms we can perform an integration by parts[2] and then (4.91) gives again only terms proportional to $Y^M{}_{RS}$ leading to the same conclusion. We therefore find the following $\Xi$-variation of the total action:

$$\begin{aligned}\delta(\Xi)\mathcal{L}_{\mathrm{VT}} \;=\;& \frac{3}{4}\varepsilon^{\mu\nu\rho\sigma}D_{RSQ}Y^Q{}_{NP}B_{\mu\nu}{}^{RS}D_\rho\Xi_\sigma{}^{NP} \\ &-\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}D_{RSQ}Y^Q{}_{NP}\Big(\partial_\mu A_\nu{}^S+\frac{1}{4}X_{TU}{}^SA_\mu{}^TA_\nu{}^U\Big)\Xi_\sigma{}^{NP}\,.\end{aligned}\tag{6.18}$$

It is clear that if the representation constraint (6.2) is satisfied, the $\Xi$-variation of the total action vanishes.

We can thus further restrict to the $\Lambda^M$ gauge transformations. According to (4.92), the $D_\rho\delta A_{\sigma\Lambda}$-term in (6.10) can then be replaced by $\frac{1}{2}\Lambda^QX_{NQ\Lambda}\mathcal{H}_{\rho\sigma}{}^N$, which can then be combined with the first term of (6.7) to form a symplectically covariant expression (the first term on the right hand side of (6.19) below). Adding also the remaining terms of (6.10) and (6.7), one obtains, using (4.106),

$$\begin{aligned}&\delta(\Lambda)\mathcal{L}_{\mathrm{VT}} =\\ &\varepsilon^{\mu\nu\rho\sigma}\Big[\frac{1}{4}\mathcal{G}_{\mu\nu}{}^M\Lambda^QX_{NQ}{}^R\Omega_{MR}\mathcal{H}_{\rho\sigma}{}^N+\frac{1}{8}\mathcal{G}_{\mu\nu}{}^M\mathcal{G}_{\rho\sigma}{}^N\Lambda^QX_{QM}{}^R\Omega_{NR}\\ &\quad+\frac{1}{4}(\mathcal{H}-\mathcal{G})_{\mu\nu\,\Lambda}Y^\Lambda{}_{NP}\Delta B_{\rho\sigma}{}^{NP}\\ &\quad-D_{MNP}A_\mu{}^MD_\nu\Lambda^N\Big(\partial_\rho A_\sigma{}^P+\frac{3}{8}X_{RS}{}^PA_\rho{}^RA_\sigma{}^S\Big)\Big]\,.\end{aligned}\tag{6.19}$$

[2]Integration by parts with the covariant derivatives is allowed as (4.88) can be read as the invariance of the tensor $X$ and (4.82) as the invariance of $\Omega$.

We observe that if the $\mathcal{H}$ in the second line was a $\mathcal{G}$, equations (4.82) and (6.12) would allow one to write the first line as an expression proportional to $D_{MNP}$. This leads to the first line in (6.20) below. The second observation is that the identity $(\mathcal{H}-\mathcal{G})^{\Lambda}=0$ allows one to rewrite the second line of (6.19) in a symplectically covariant way, so that, altogether, we have

$$\begin{aligned}
&\delta(\Lambda)\mathcal{L}_{\rm VT} = \\
&\quad \varepsilon^{\mu\nu\rho\sigma}\left[\frac{1}{4}\mathcal{G}_{\mu\nu}{}^{M}\Lambda^{Q}X_{NQ}{}^{R}\Omega_{MR}(\mathcal{H}-\mathcal{G})_{\rho\sigma}{}^{N}+\frac{3}{8}\mathcal{G}_{\mu\nu}{}^{M}\mathcal{G}_{\rho\sigma}{}^{N}\Lambda^{Q}D_{QMN}\right.\\
&\qquad -\frac{1}{4}(\mathcal{H}-\mathcal{G})_{\mu\nu}{}^{M}\Omega_{MR}Y^{R}{}_{NP}\Delta B_{\rho\sigma}{}^{NP}\\
&\qquad \left.-D_{MNP}A_{\mu}{}^{M}D_{\nu}\Lambda^{N}\left(\partial_{\rho}A_{\sigma}{}^{P}+\frac{3}{8}X_{RS}{}^{P}A_{\rho}{}^{R}A_{\sigma}{}^{S}\right)\right]. \qquad (6.20)
\end{aligned}$$

By choosing $\Delta B_{\rho\sigma}{}^{NP}$ as in (4.107), the result (6.20) becomes

$$\begin{aligned}
&\delta(\Lambda)\mathcal{L}_{\rm VT} = \\
&\quad \varepsilon^{\mu\nu\rho\sigma}\left[\frac{3}{8}\Lambda^{Q}D_{MNQ}\left(2\mathcal{G}_{\mu\nu}{}^{M}(\mathcal{H}-\mathcal{G})_{\rho\sigma}{}^{N}+\mathcal{G}_{\mu\nu}{}^{M}\mathcal{G}_{\rho\sigma}{}^{N}\right)\right.\\
&\qquad \left.-D_{MNP}A_{\mu}{}^{M}D_{\nu}\Lambda^{N}\left(\partial_{\rho}A_{\sigma}{}^{P}+\frac{3}{8}X_{RS}{}^{P}A_{\rho}{}^{R}A_{\sigma}{}^{S}\right)\right], \qquad (6.21)
\end{aligned}$$

which is then again proportional to $D_{MNP}$, and hence zero when the original representation constraint (6.2) is imposed.

Our goal in the remainder of this chapter is to generalize this result for theories with quantum anomalies. We will see how the formalism needs to be modified and in particular, how an adjusted version of the representation constraint restores the invariance of the full quantum effective action via a Green-Schwarz cancellation mechanism.

## 6.3 Gauge invariance of the effective action with anomalies

#### Symplectically covariant anomalies

In §4.3 we discussed the algebraic constraints that are imposed on the embedding tensor and in the previous section, we showed how they play a crucial role in the construction of a gauge invariant Lagrangian with electric and magnetic gauge potentials as well as tensor fields. Two of these constraints, (4.85) and (4.111), had a very clear physical motivation and ensured the closure of the gauge algebra and the mutual locality of all interacting fields. The physical origin of the third constraint, the representation constraint, (4.105), on the other hand, remained a bit obscure. In order to understand its meaning, we specialize it to its purely electric components:

$$X_{(\Lambda\Sigma\Omega)} = 0\,. \tag{6.22}$$

Given that the tensors $X_{\Lambda\Sigma\Omega}$ generate axionic shift symmetries (remember the first term on the right hand side of (4.83)), we can identify them with the corresponding symbols $X_{\Lambda\Sigma\Omega}$ in chapter 5, and recognize (6.22) as the condition for the absence of quantum anomalies for the electric gauge bosons (see (5.74)). It is therefore suggestive to interpret the representation constraint (4.105) as the condition for the absence of quantum anomalies for *all* gauge fields (both electric and magnetic), and one expects that in the presence of quantum anomalies, this constraint can be relaxed. We will show that the relaxation consists in assuming that the symmetric tensor $D_{MNP}$ is of the form[3]

$$D_{MNP} = d_{MNP}\,, \tag{6.23}$$

for a symmetric tensor $d_{MNP}$ which describes the quantum gauge anomalies due to anomalous chiral fermions:

$$\delta(\Lambda)\Gamma[A] = \int \mathrm{d}^4x\, \Lambda^M \mathcal{A}_M\,, \tag{6.24}$$

with

$$\begin{aligned}\mathcal{A}_M \quad = \quad & -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\Lambda^P d_{MNP}\partial_\mu A_\nu{}^N \partial_\rho A_\sigma{}^P \qquad (6.25)\\ & -\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\Lambda^P \left(d_{NPR}X_{[MS]}{}^P + \frac{3}{2}d_{MNP}X_{[RS]}{}^N\right)\partial_\mu A_\nu{}^N A_\rho{}^R A_\sigma{}^S\,.\end{aligned}$$

This expression formally looks like a symplectically covariant generalization of the electric consistent anomaly (5.71).

In fact, one expects the anomalies $\mathcal{A}_M$ from the loops of those fermions, $\psi$, that interact with the gauge fields via minimal couplings

$$\bar{\psi}\gamma^\mu(\partial_\mu - A_\mu{}^\Lambda\Theta_\Lambda{}^\alpha\delta_\alpha - A_{\mu\Lambda}\Theta^{\Lambda\alpha}\delta_\alpha)\psi\,. \tag{6.26}$$

Therefore, the anomalies contain – for each external gauge field (or gauge parameter) – an embedding tensor, i.e. $d_{MNP}$ has the following particular form:

$$d_{MNP} = \Theta_M{}^\alpha\Theta_N{}^\beta\Theta_P{}^\gamma d_{\alpha\beta\gamma}\,, \tag{6.27}$$

[3]The possibility to impose a relation such as (6.23) is by no means guaranteed for all types of gauge groups (see e.g. [114] for a short discussion in the purely electric case studied in [1]).

with $d_{\alpha\beta\gamma}$ being a constant symmetric tensor (i.e., independent of the scalar fields). We expect this constancy to be generally true for the same topological reasons that imply the constancy of $d_{\Lambda\Gamma\Omega}$ in the conventional electric gaugings [97, 98]. In the familiar context of a theory with a flat scalar manifold, constant fermionic transformation matrices, $t_\alpha$, and the corresponding minimal couplings, the tensor $d_{\alpha\beta\gamma}$ is simply proportional to

$$d_{\alpha\beta\gamma} \propto \mathrm{Tr}(\{t_\alpha, t_\beta\} t_\gamma), \tag{6.28}$$

where the trace is over the representation matrices of the fermions.[4]

We will now see how the anomaly (6.24) can be canceled by a classical variation of $\mathcal{L}_{\mathrm{VT}}$ if we implement the modified representation constraint (6.23). In other words, we will discuss the conditions under which

$$\delta(\Lambda, \Xi)\left(\Gamma[A] + S_{\mathrm{VT}}\right) = 0\,. \tag{6.29}$$

### Anomaly cancellation

In order to check (6.29), one can use the results from sections 6.2 - 6.2 and make the replacement $D_{MNP} = d_{MNP}$ everywhere. However, the variations (6.18) and (6.21) reveal two problems that need to be solved:

(i) A priori, the $\Xi$-variation in (6.18) does not vanish.

(ii) The $\Lambda$-variation in (6.21) depends on the scalar fields via the field strengths $\mathcal{G}$, whereas the anomaly does not.

Let us first solve item number (i). It is immediately clear form (6.18) that $D_{RSQ}$ – and hence $d_{RSQ}$ if we use the identification (6.23) – is always contracted with a tensor $Y^Q{}_{NP}$. Then one can use the special form of the anomaly tensor $d_{RSQ}$ in (6.27) to show that

$$d_{RSQ} Y^Q{}_{NP} = d_{\alpha\beta\gamma} \Theta_R{}^\alpha \Theta_S{}^\beta \Theta_Q{}^\gamma Y^Q{}_{NP} = 0\,. \tag{6.30}$$

The last equality is an immediate consequence of the quadratic constraint, i.e., the contraction of an embedding tensor with $Y^Q{}_{NP}$ vanishes (recall equation (4.87)). This solves our first problem since we now have that $\delta(\Xi)\mathcal{L}_{\mathrm{VT}} = 0$.

The second issue requires some more work. Indeed, in order to obtain a variation $\delta(\Lambda)\mathcal{L}_{\mathrm{VT}}$ that does not depend on $\mathcal{G}$, we have to replace $\Delta B_{\rho\sigma}{}^{NP}$ in the original transformation of the two-form, (4.107), by a new expression such that

$$Y^R{}_{NP} \Delta B_{\rho\sigma}{}^{NP} = -2 Y^R{}_{NP} \Lambda^N \mathcal{G}_{\rho\sigma}{}^P + \frac{3}{2} \Omega^{RM} d_{MNQ} \Lambda^Q (\mathcal{H} - \mathcal{G})_{\rho\sigma}{}^N\,. \tag{6.31}$$

[4] One might wonder how the magnetic vector fields $A_{\mu\Lambda}$ can give rise to anomalous triangle diagrams, as they have no propagator due to the lack of a kinetic term. However, it is the *amputated* diagram with internal fermion lines that one has to consider.

Inserting this in (6.20) would lead to

$$\begin{aligned}\delta(\Lambda)\mathcal{L}_{\rm VT} &= \varepsilon^{\mu\nu\rho\sigma}\left[\frac{3}{8}\Lambda^Q d_{MNQ}\mathcal{F}_{\mu\nu}{}^M\mathcal{F}_{\rho\sigma}{}^N\right. \\ &\qquad \left. -d_{MNP}A_\mu{}^M D_\nu\Lambda^N\left(\partial_\rho A_\sigma{}^P+\frac{3}{8}X_{RS}{}^P A_\rho{}^R A_\sigma{}^S\right)\right],\end{aligned} \tag{6.32}$$

where we have used (6.30) to delete contributions coming from the $B_{\mu\nu}{}^{NP}$ term in $\mathcal{H}_{\mu\nu}{}^M$. The new variation (6.32) does not depend on the scalar fields anymore.

Let us pause for a while and study the properties of the new transformation (6.31). The first term on the right hand side would follow from (4.107), but the second term cannot in general be obtained from assigning transformations to $B_{\rho\sigma}{}^{NP}$ (compare with (6.12)). Indeed, self-consistency of (6.31) requires that the second term on the right hand side be proportional to $Y^R{}_{NP}$, which imposes a further constraint on $d_{MNP}$ (or $D_{MNP}$). We will see in §6.3 how we can nevertheless justify the transformation law (6.31) by introducing other antisymmetric tensors. For the moment, we just accept (6.31) and explore its consequences.

Expanding (6.32) using (4.90) and (4.84), and doing a partial integration, the variation in (6.32) can be rewritten as

$$\delta(\Lambda)S_{\rm VT}=\delta(\Lambda)\int \mathrm{d}^4x\,\mathcal{L}_{\rm VT}=-\int \mathrm{d}^4x\,\Lambda^M\mathcal{A}_M\,. \tag{6.33}$$

If we combine this with (6.24), we find the desired result (6.29).

Let us summarize the result of our calculation up to the present point. We have used the action (6.11) and considered its transformations under (4.93) and (4.106), where $\Delta B_{\mu\nu}{}^{NP}$ was undetermined. We showed that the choice (4.107) leads to invariance if the closure and (original) representation constraints are satisfied. However, when we use instead the more general transformation (6.31) in the case $D_{MNP}=d_{MNP}$, we obtain the non-vanishing classical variation (6.33). This corresponds exactly to a symplectically covariant generalization of the electric consistent quantum anomaly, but with the opposite sign. Therefore, we have shown that the classical variation of the action $S_{\rm VT}$ cancels the quantum anomaly.

In order to fully justify and understand this result, we are then left with one open issue; we have to show how the transformation (6.31), which underlies the result (6.33), can be realized. This will be done in the next section.

### New antisymmetric tensors

The goal of this section is to justify the transformation (6.31), without requiring further constraints on the $D$-tensor. That transformation gives an expression for

$Y^R{}_{NP}\Delta B_{\rho\sigma}{}^{NP}$ that is not obviously a contraction with the tensor $Y^R{}_{NP}$. We can therefore in general not assign a transformation to $B_{\rho\sigma}{}^{NP}$ such that its contraction with $Y^R{}_{NP}$ gives (6.31). To overcome this problem, we will have to change the set of independent antisymmetric tensors. The $B_{\mu\nu}{}^{MN}$ cannot be considered as independent fields in order to realize (6.31). We will, as in [61], introduce a new set of independent antisymmetric tensors, denoted by $B_{\mu\nu\,\alpha}$ for any $\alpha$ denoting a rigid symmetry.

The fields $B_{\mu\nu}{}^{NP}$ and their associated gauge parameters $\Xi_\mu{}^{NP}$ appeared in the relevant formulae in the form $Y^M{}_{NP}B_{\mu\nu}{}^{NP}$ or $Y^M{}_{NP}\Xi_\mu{}^{NP}$, see e.g. (4.93), (4.97) and (4.104). Using (6.12), (6.15) and (6.27) it follows that $Y^P{}_{MN}$ can be written as

$$Y^P{}_{MN} = Z^{P\alpha}\tilde{\Delta}_{\alpha MN}\,, \tag{6.34}$$

with

$$\tilde{\Delta}_{\alpha MN} \equiv \Delta_{\alpha NP} - 3d_{\alpha\beta\gamma}\Theta_N{}^\beta\Theta_P{}^\gamma\,, \tag{6.35}$$

and therefore

$$Y^M{}_{NP}B_{\mu\nu}{}^{NP} = Z^{M\alpha}\tilde{\Delta}_{\alpha MN}B_{\mu\nu}{}^{MN}\,. \tag{6.36}$$

We will therefore replace the tensors $B_{\mu\nu}{}^{MN}$ by new tensors $B_{\mu\nu\,\alpha}$ using

$$\tilde{\Delta}_{\alpha MN}B_{\mu\nu}{}^{MN} \quad \rightarrow \quad B_{\mu\nu\,\alpha}\,. \tag{6.37}$$

and consider the $B_{\mu\nu\,\alpha}$ as the independent antisymmetric tensors. There is thus one tensor for every generator of the rigid symmetry group. In particular, the replacement (6.37) implies that

$$Y^M{}_{NP}B_{\mu\nu}{}^{NP} \quad \rightarrow \quad Z^{M\alpha}B_{\mu\nu\,\alpha}\,. \tag{6.38}$$

We also introduce a corresponding set of independent gauge parameters $\Xi_{\mu\,\alpha}$ through the substitution:

$$\tilde{\Delta}_{\alpha MN}\Xi_\mu{}^{MN} \quad \rightarrow \quad \Xi_{\mu\,\alpha}\,. \tag{6.39}$$

This allows us to reformulate all the equations in the previous sections in terms of $B_{\mu\nu\,\alpha}$ and $\Xi_{\mu\,\alpha}$. For instance we will write:

$$\begin{aligned}
\delta A_\mu{}^M &= D_\mu\Lambda^M - Z^{M\alpha}\Xi_{\mu\,\alpha}\,, &(6.40)\\
\mathcal{H}_{\mu\nu}{}^M &= \mathcal{F}_{\mu\nu}{}^M + Z^{M\alpha}B_{\mu\nu\,\alpha}\,, &(6.41)\\
\mathcal{L}_{\mathrm{top},B} &= \frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\,Z^{\Lambda\alpha}B_{\mu\nu\,\alpha}\Big(\mathcal{F}_{\rho\sigma\,\Lambda} + \frac{1}{2}\,Z_\Lambda{}^\alpha B_{\mu\nu\,\alpha}\Big)\,. &(6.42)
\end{aligned}$$

Moreover, we can now also define

$$\begin{aligned}
\delta B_{\mu\nu\,\alpha} &= 2\,D_{[\mu}\Xi_{\nu]\alpha} + 2\tilde{\Delta}_{\alpha\,NP}A_{[\mu}{}^N\delta A_{\nu]}{}^P + \Delta B_{\mu\nu\,\alpha}\,,\\
\Delta B_{\mu\nu\,\alpha} &= -2\tilde{\Delta}_{\alpha NP}\Lambda^N\mathcal{G}_{\mu\nu}{}^P + 3d_{\alpha\beta\gamma}\Theta_N{}^\beta\Theta_P{}^\gamma\Lambda^N(\mathcal{H}-\mathcal{G})_{\mu\nu}{}^P\,, &(6.43)
\end{aligned}$$

to reproduce (6.31), where the left-hand side of (6.31) is replaced according to (6.38). Here the covariant derivative is defined as

$$D_{[\mu}\Xi_{\nu]\alpha} = \partial_{[\mu}\Xi_{\nu]\,\alpha} + f_{\alpha\beta}{}^{\gamma}\Theta_P{}^{\beta}A_{[\mu}{}^{P}\Xi_{\nu]\,\gamma}\,. \tag{6.44}$$

To further check the consistency of our results, we will in the next section reduce our treatment to a purely electric gauging and show that the results of chapter 5 can be reproduced.

**Purely electric gaugings**

Let us first explicitly write down $D_{MNP}$ in its electric and magnetic components:

$$\begin{aligned} D_{\Lambda\Sigma\Gamma} &= X_{(\Lambda\Sigma\Gamma)}\,, \\ 3D^{\Lambda}{}_{\Sigma\Gamma} &= X^{\Lambda}{}_{\Sigma\Gamma} - 2Y^{\Lambda}{}_{\Sigma\Gamma}\,, \\ 3D^{\Lambda\Sigma}{}_{\Gamma} &= -X_{\Gamma}{}^{\Lambda\Sigma} + 2Y_{\Gamma}{}^{\Lambda\Sigma}\,, \\ D^{\Lambda\Sigma\Gamma} &= -X^{(\Lambda\Sigma\Gamma)}\,. \end{aligned} \tag{6.45}$$

In the case of a purely electric gauging, the only non-vanishing components of the embedding tensor are electric:

$$\Theta_M{}^{\alpha} = (\Theta_{\Lambda}{}^{\alpha},\ 0)\ . \tag{6.46}$$

Therefore also $X^{\Lambda}{}_{N}{}^{P} = 0$ and (6.27) implies that the only non-zero components of $D_{MNP} = d_{MNP}$ are $D_{\Lambda\Sigma\Omega}$. Therefore, (6.45) reduce to

$$D_{\Lambda\Sigma\Omega} = X_{(\Lambda\Sigma\Omega)}\,, \qquad X_{(\Sigma\Omega)}{}^{\Lambda} = 0\,, \qquad X_{\Omega}{}^{\Lambda\Sigma} = 0\,. \tag{6.47}$$

The non-vanishing entries of the gauge generators are $X_{\Lambda\Sigma\Gamma}$ and $X_{\Sigma\Omega}{}^{\Lambda} = -X_{\Sigma}{}^{\Lambda}{}_{\Omega} = X_{[\Sigma\Omega]}{}^{\Lambda}$, the latter satisfying the Jacobi identities since the right hand side of (4.89) for $MNQR$ all electric indices vanishes. The $X_{[\Sigma\Omega]}{}^{\Lambda}$ can be identified with the structure constants of the gauge group that were introduced e.g. in (4.42). The $X_{\Lambda\Sigma\Omega}$ correspond to the shifts in (4.48). The first relation in (6.47) then corresponds to (5.74).

The locality constraint is trivially satisfied and the closure relation reduces to (4.51) as expected.

At the level of the action $\mathcal{L}_{\mathrm{VT}}$, all tensor fields drop out since, when we express everything in terms of the new tensors $B_{\mu\nu\,\alpha}$, these tensors always appear contracted with a factor $\Theta^{\Lambda\,\alpha} = 0$. In particular, the topological terms $\mathcal{L}_{\mathrm{top},B}$

vanish and the modified field strengths for the electric vector fields $\mathcal{H}_{\mu\nu}{}^{\Lambda}$ reduce to ordinary field strengths:

$$\mathcal{H}_{\mu\nu}{}^{\Lambda} = 2\partial_{[\mu}A_{\nu]}{}^{\Lambda} + X_{[\Omega\Sigma]}{}^{\Lambda}A_{\mu}{}^{\Omega}A_{\nu}{}^{\Sigma} . \qquad (6.48)$$

Also the GCS terms (4.103) reduce to their purely electric form (5.61) with $X^{(\mathrm{CS})}_{\Omega\Lambda\Sigma} = X^{(\mathrm{m})}_{\Omega\Lambda\Sigma}$. Finally, the gauge variation of $\mathcal{L}_{\mathrm{VT}}$ reduces to minus the ordinary consistent gauge anomaly, as we presented it in (5.71).

This concludes our reinvestigation of the electric gauging with axionic shift symmetries, GCS terms and quantum anomalies as it follows from our more general symplectically covariant treatment. We showed that the more general theory reduces consistently to the known case of a purely electric gauging.

## 6.4 Simple example

Let us now briefly illustrate the results from this chapter by means of a simple example. We consider a theory with a rigid symmetry group embedded in the electric/magnetic duality group $Sp(2,\mathbb{R})$. The embedding in the symplectic transformations is given by

$$t_{1M}{}^{N} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad t_{2M}{}^{N} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \qquad t_{3M}{}^{N} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad (6.49)$$

i.e. $t_2{}^{11} = 1$. Let us consider the following subset of duality transformations:

$$\mathcal{S}^{M}{}_{N} = \delta^{M}{}_{N} - \Lambda^{P} X_{PN}{}^{M} , \quad \text{with generators} \quad X_{PM}{}^{N} = \begin{pmatrix} 0 & 0 \\ X_{P}{}^{11} & 0 \end{pmatrix} , \quad (6.50)$$

where $\Lambda^P$ is the rigid transformation parameter. The tensor $X$ is related to the embedding of the symmetries in the symplectic algebra using the embedding tensor,

$$X_{PM}{}^{N} = \sum_{\alpha=1}^{3} \Theta_{P}{}^{\alpha} t_{\alpha M}{}^{N} . \qquad (6.51)$$

We have thus chosen the embedding tensor

$$\Theta_{P}{}^{1} = 0 , \qquad \Theta_{P}{}^{2} = X_{P}{}^{11} , \qquad \Theta_{P}{}^{3} = 0 . \qquad (6.52)$$

We now want to promote $\mathcal{S}^{M}{}_{N}$ to be a gauge transformation, i.e., we take the $\Lambda^N = \Lambda^N(x)$ spacetime dependent and the $X_{PM}{}^{N}$ are the gauge generators. This obviously corresponds to a magnetic gauging, as (6.47) is violated, and therefore requires the formalism that was developed in [61] and reviewed in §4.3. Closure of the gauge algebra spanned by the $X_{PM}{}^{N}$ requires that we impose (4.85), where only the right-hand side is non-trivial. It requires $\Theta_1{}^2 = 0$, and thus the only gauge generators that are consistent with this constraint are

$$X_{PM}{}^N = (X_{1M}{}^N,\ X^1{}_M{}^N)\,, \quad \text{with} \quad X_{1M}{}^N = 0\,, \quad X^1{}_M{}^N = \begin{pmatrix} 0 & 0 \\ X^{111} & 0 \end{pmatrix}. \tag{6.53}$$

Note that this choice still violates the original linear representation constraint (6.2), as (6.45) gives $D^{111} = -X^{111} \neq 0$. However, this does not prevent us from performing the gauging with generators $X_{PM}{}^N$ given in (6.53). We introduce a vector $A_\mu{}^M$ which contains an electric and a magnetic part, $A_\mu{}^1$ and $A_{\mu 1}$. Note that only the magnetic vector couples to matter via covariant derivatives since the embedding tensor projects out the electric part. In what follows, we also assume the presence of anomalous couplings between the magnetic vector and chiral fermions. As we will now review, this justifies the nonzero $X^{111} \neq 0$, since it will give rise to anomaly cancellation terms in the classical gauge variation of the action. More precisely, we will have to require that

$$\Theta^{12} = X^{111}\,, \qquad -X^{111} = d^{111} = (X^{111})^3 \tilde{d}_{222}\,, \tag{6.54}$$

where we introduced $\tilde{d}_{222}$ as the component of $d_{\alpha\beta\gamma}$.

To show this, we first introduce a kinetic term for the electric vector fields:

$$\mathcal{L}_{\text{g.k.}} = \frac{1}{4}\, e\, \mathcal{I}\, \mathcal{H}_{\mu\nu}{}^1 \mathcal{H}^{\mu\nu\, 1} - \frac{1}{8}\, \mathcal{R}\, \varepsilon^{\mu\nu\rho\sigma} \mathcal{H}_{\mu\nu}{}^1 \mathcal{H}_{\rho\sigma}{}^1, \tag{6.55}$$

where we introduced the modified field strength (6.41)

$$\mathcal{H}_{\mu\nu}{}^1 = 2\partial_{[\mu} A_{\nu]}{}^1 + \frac{1}{2} X^{111} B_{\mu\nu 2}\,, \tag{6.56}$$

which depends on a tensor field $B_{\mu\nu 2}$ and therefore transforms covariantly under

$$\begin{aligned} \delta A_\mu{}^1 &= \partial_\mu \Lambda^1 + X^{111} A_{\mu\, 1} \Lambda_1 - \frac{1}{2} X^{111} \Xi_{\mu 2}\,, \\ \delta B_{\mu\nu 2} &= 2\partial_{[\mu} \Xi_{\nu]2} + 4 A_{[\mu\, 1} \partial_{\nu]} \Lambda_1 - 6 \Lambda_1 \partial_{[\mu} A_{\nu]\, 1} - \Lambda_1 \mathcal{G}_{\mu\nu\, 1}\,, \\ \delta A_{\mu 1} &= \partial_\mu \Lambda_1\,. \end{aligned} \tag{6.57}$$

This follows from (6.43) since the only nonzero component of $\Delta_{2MN}$ is $\Delta_2{}^{11} = 2$ and for $d_{2MN}$ we have only $d_2{}^{11} = -1$. One can check that

$$\delta \mathcal{H}_{\mu\nu}{}^1 = -\frac{1}{2} X^{111} \Lambda_1 (\mathcal{H} + \mathcal{G})_{\mu\nu\, 1}\,, \quad \text{with} \tag{6.58}$$

$$\mathcal{H}_{\mu\nu\, 1} = \mathcal{F}_{\mu\nu\, 1} = 2\partial_{[\mu} A_{\nu]1}\,, \qquad \mathcal{G}_{\mu\nu\, 1} \equiv \mathcal{R} \mathcal{H}_{\mu\nu}{}^1 + \frac{1}{2} e \mathcal{I} \varepsilon_{\mu\nu\rho\sigma} \mathcal{H}^{\rho\sigma\, 1}\,.$$

Under gauge variations, the real and imaginary part of the kinetic function transform as follows (cf. (4.83)):

$$\delta\mathcal{I} = 2\Lambda_1 X^{111}\mathcal{R}\mathcal{I}\,, \qquad \delta\mathcal{R} = \Lambda_1 X^{111}\left(\mathcal{R}^2 - \mathcal{I}^2\right)\,. \tag{6.59}$$

Then it's a short calculation to show that

$$\delta\mathcal{L}_{\text{g.k.}} = \frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\Lambda_1 X^{111}\mathcal{G}_{\mu\nu\,1}\partial_\rho A_{\sigma 1}\,. \tag{6.60}$$

This is consistent with (6.7).

In a second step, we add the topological term (6.42):

$$\mathcal{L}_{\text{top},B} = \frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}X^{111}B_{\mu\nu 2}\partial_{[\rho}A_{\sigma]\,1}\,. \tag{6.61}$$

The gauge variation of this term is equal to (up to a total derivative)

$$\delta\mathcal{L}_{\text{top},B} = -\frac{1}{4}\Lambda_1 X^{111}\varepsilon^{\mu\nu\rho\sigma}\left(\partial_\mu A_{\nu\,1}\right)\left(2\partial_\rho A_{\sigma\,1} + \mathcal{G}_{\rho\sigma\,1}\right)\,. \tag{6.62}$$

The generalized Chern-Simons term (4.103) vanishes in this case. Combining (6.60) and (6.62), one derives

$$\delta\left(\mathcal{L}_{\text{g.k.}} + \mathcal{L}_{\text{top},B}\right) = -\frac{1}{2}\Lambda_1 X^{111}\left(\partial_\mu A_{\nu\,1}\right)\left(\partial_\rho A_{\sigma\,1}\right)\varepsilon^{\mu\nu\rho\sigma}\,. \tag{6.63}$$

This cancels the magnetic gauge anomaly whose form can be derived from (6.25):

$$\mathcal{A}[\Lambda] \quad = \quad -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\Lambda_1 d^{111}\left(\partial_\mu A_{\nu\,1}\right)\left(\partial_\rho A_{\sigma\,1}\right)\,, \tag{6.64}$$

if we remember that $X^{111} = -D^{111} = -d^{111}$. Note that the electric gauge fields do not appear which corresponds to the fact that the electric gauge fields do not couple to the chiral fermions.

A simple fermionic spectrum that could yield such an anomaly (6.64) is given by, e.g., three chiral fermions with canonical kinetic terms and quantum numbers $Q = (-1), (-1), (+2)$ under the $U(1)$ gauged by $A_{\mu\,1}$. Indeed, with this spectrum, we would have $\mathrm{Tr}(Q) = 0$, i.e., vanishing mixed gauge-gravitational anomaly, but a cubic Abelian gauge anomaly $d^{111} \propto \mathrm{Tr}(Q^3) = +6$.

## 6.5 Conclusions

In this chapter we have shown how general gauge theories with axionic shift symmetries, generalized Chern-Simons terms and quantum anomalies can be formulated in a way that is covariant with respect to electric/magnetic duality transformations. This generalizes previous work of [61], in which only *classically* gauge invariant theories with anomaly-free fermionic spectra are considered. Whereas the focus in [61] was on extended (and hence automatically anomaly-free) gauged supergravity theories, our results here can be applied to general $\mathcal{N} = 1$ gauged supergravity theories with possibly anomalous fermionic spectra. Such anomalous fermionic spectra are a natural feature of many string compactifications, notably of intersecting brane models in type II orientifold compactifications, where also GCS terms frequently occur [10]. Especially in combination with background fluxes, such compactifications may naturally lead to $4D$ actions with tensor fields and gaugings in unusual duality frames. Our formulation accommodates all these non-standard formulations, just as ref. [61] does in the anomaly-free case.

At a technical level, our results were obtained by relaxing the so-called representation constraint to allow for a symmetric three-tensor $d_{MNP}$ that parameterizes the quantum anomaly. In contrast to the other constraints for the embedding tensor, this modified representation constraint is not homogeneous in the embedding tensor, which is a novel feature in this formalism. Also our treatment gave an interpretation for the physical meaning of the representation constraint: in its original form used in [61], it simply states the absence of quantum anomalies. It is interesting, but in retrospect not surprising, that the extended supergravity theories from which the original constraint has been derived in [61], need this constraint for their internal classical consistency.

It would be interesting to embed our results in a manifestly supersymmetric framework. Likewise, it would be interesting to study explicit $\mathcal{N} = 1$ string compactifications within the framework used in this paper, making use of manifest duality invariances. Another topic we have not touched upon are Kähler anomalies [87–90, 122–126] in $\mathcal{N} = 1$ supergravity, or gravitational anomalies.

Chapter 7

# Generalized gaugings and the field-antifield formalism

*This final chapter provides an overview of the research results that were obtained in collaboration with Frederik Coomans and Antoine Van Proeyen in [3]. We investigate the gauge structure of the embedding tensor formalism and show that it generically involves open, soft and reducible algebra's. These are the properties for which the field-antifield formalism was constructed and we demonstrate how the embedding tensor formalism can be connected to it.*

## 7.1 Introduction

The quantization of a general gauge theory using the path integral formalism leads to two complications that are both related to the degeneracy of the physical content in the gauge theory. The first problem finds its origin in the anomalous transformation of the path integral measure, and signals the breakdown of local invariance in the quantum theory. Put differently, the physical degeneracy is lost at the quantum level, and the quantum theory contains unphysical (negative-norm) states that cannot be eliminated. In order to avoid this problem, one has to demand anomaly-freedom which imposes strong constraints on the types of gauge groups and/or the physical content of the theory. For example, in the case of $\mathcal{N} = 1$ supersymmetric gauge theories these constraints have been discussed in previous chapters, together with a Green-Schwarz cancellation mechanism that might eliminate some of the constraints. For simplicity, though, we will assume the *absence of quantum anomalies* in this chapter. Instead, we will focus on a second difficulty, which is the gauge fixing procedure.

The path integral contains a functional integral over all physical field configurations between a given initial state and final state. In particular, one has to integrate over the gauge fields. However, this is not straightforward since the gauge field configurations that are related by gauge transformations are physically equivalent. Therefore a straightforward integration over $A_\mu$ involves the integration over physically equivalent configurations and leads to an infinite result for the path integral. The way to solve this problem is to fix a particular gauge for the $A_\mu$-fields, before carrying out the integration. In the ideal case, we would like to perform this gauge fixing procedure without breaking the (space-time) symmetries in the theory. For Abelian theories, this is long understood, whereas for Yang-Mills theories all the issues were resolved in the work of L.D. Faddeev and V.N. Popov [127]. Their solution involves unphysical ghost fields (they violate the spin-statistics theorem) that compensate for the effects of the gauge degrees of freedom. These ghost fields arise as follows. First one inserts a (gauge invariant) unity factor in the path integral,

$$\mathbb{1} = \int [d\theta(x)]\delta(G(A^\theta))\det\left(\frac{\delta G(A^\theta)}{\delta\theta}\right) , \tag{7.1}$$

where $G(A)$ is some function we wish to set equal to zero as a gauge fixing condition (for example $G(A) = \partial^\mu A_\mu$ corresponds to the Lorentz gauge), and $A^\theta_\mu$ denotes the transformed gauge field: $A^\theta_\mu = A_\mu + \partial_\mu\theta(x)$. Then after some rewriting we discover the equivalence between two modified path integrals. On the one side, we have a path integral that contains a delta function $\delta(G(A^\theta))$ which restricts the functional integral $\int[dA]$ to physically inequivalent configurations. Equivalently, the delta function can be replaced by a Jacobian measure factor that devides out the volume of the gauge transformations in function space. It is this factor that can be rewritten as a functional integral over ghost fields, with extra ghost terms added to the original Lagrangian.

Despite its success for ordinary Yang-Mills gauge theories, the Faddeev-Popov formalism does not directly apply to theories with a more complicated gauge structure. These include gauge theories with an open, soft and/or reducible gauge algebra.[1] These are exactly the types of gauge groups that arise naturally in

[1]The properties of these algebras are as follows. *Open* means that the commutators of two gauge transformations on a field only close up to terms that are proportional to the equations of motion of the same field. *Soft* algebra's arise when the structure constants are not really constant, but depend on the fields and parameters in the theory. We will use the more correct terminology of "structure functions" in this case. Finally, for a *reducible* algebra, the gauge transformations are not all independent.

supergravity constructions, due to the presence of higher order form fields, the use of non-supersymmetric gauges (such as the Wess-Zumino gauge) and the presence of off-shell multiplets. But most importantly for us, open, soft and reducible algebras are also generically present in the embedding tensor formalism (a detailed proof will be presented in §7.3).

In order to deal with these complicated properties in a quantization procedure, it is necessary to add extra ingredients to the original Faddeev-Popov method. The details of this extension were first identified and analyzed by Batalin and Vilkovisky in [128–132] and more details can be found e.g. in [133, 134]. The general structure is as follows.

1. For every gauge parameter in the algebra, we introduce a ghost field. For every zero mode in the (reducible) algebra, we introduce a ghost for ghost field, etc.

2. Each of the fields in 1. gets a corresponding antifield which should be regarded as a mathematical tool to set up the formalism.[2]

3. There exists an odd symplectic form on the space of fields and antifields which is called the antibracket and will be denoted by $(\cdot, \cdot)$. The original (gauge invariant) classical action $S_0$ will be extended to a new action $S$ that is an expansion in the antifields.[3] This extended action is essentially unique if one imposes the classical master equation $(S, S) = 0$. This equation reproduces in a compact way the entire gauge structure of the original theory that is governed by $S_0$.

All these ingredients make the Batalin-Vilkovisky (BV) formalism more general than the Faddeev-Popov approach. In fact, it is currently the most powerful method for quantizing gauge theories.

Besides being a powerful tool for gauge theory quantization, it should be noted that the BV formalism is also very well suited for the concise description of *classical* theories with a complicated gauge structure. The reason is that this formalism introduces ghost fields from the outset, and the extended action $S$ –which is a functional of all the fields and antifields– incorporates the full gauge structure via the master equation. It is this property that will be most important for our work. Indeed, we will exhaust the capacity of the BV formalism to describe the

[2]Due to the presence of fields and antifields, the Batalin-Vilkovisky formalism will often be called the field-antifield formalism in the literature.

[3]The extended action is not yet suited for the use in a path integral since it possesses gauge invariances. These need to be fixed first, which is typically done via a gauge fixing condition that allows one to write the antifields in terms of the fields. The details of this procedure are beyond the scope of this text and can be found e.g. in [134].

complicated gauge structure of the embedding tensor formalism in a concise way, leaving the actual quantization for future research.

The outline of this chapter is as follows. In §7.2 we will introduce some necessary notations and conventions. These include a more precise identification of the 2-forms in the embedding tensor formalism, and an overview of the deWitt notation. Once we have this machinery at our disposal, we will start our investigation of the gauge structures that characterize the embedding tensor formalism. Recall from §4.3 that we have two types of gauge transformations; the electromagnetic gauge transformations of the 1-forms with parameters $\Lambda^M$, and shift transformations of the 1-forms with parameters $\Xi_\mu{}^{MN}$. In Table 7.1 we provide a schematic overview of these ingredients. Note that we have made a distinction between the formalism

**Table 7.1:** Schematic overview of the embedding tensor formalism.

| | formalism without an action | formalism with an invariant action |
|---|---|---|
| constraints on $\Theta_M{}^\alpha$: | closure constraint | closure and linear constraint |
| field content: | $A_\mu{}^M$, $B_{\mu\nu}{}^{MN}$ | $A_\mu{}^M$, $B_{\mu\nu}{}^{MN}$ |
| gauge transformations: | $\delta A = D\Lambda - \Theta\Xi$, | $\delta A = D\Lambda - \Theta\Xi$, |
| | $\delta B = D\Xi + \dots$ | $\delta B = D\Xi + \Delta B + \dots$ |

in the absence and in the presence of a gauge invariant action. In principle, one does not have to make this distinction because the only relevant case is the one with a dynamical description for the fields. Nevertheless, we will often study the properties in the much easier case without an action first, and later extend the results to the formalism in the presence of an action.

There are two essential differences between these two cases (i.e. the second and third column in Table 7.1). First one should notice that the linear constraint on the embedding tensor is only introduced to show gauge invariance of the action (recall §4.3) and therefore, it can be omitted in our discussion about the formalism in the absence of an action. The second difference is contained in the gauge transformation of the 2-forms. In §4.3 we have motivated the presence of an extra term $\Delta B_{\mu\nu}{}^{MN}$ which was given by the expression in (4.107) – at least in the absence of quantum anomalies.

For completeness, we should also mention that the second column in Table 7.1 can be extended beyond the 2-tensors, by adding extra $p$-forms ($p > 2$) and extra gauge transformations. This construction leads to the so-called "tensor hierarchy" [74, 85, 135–137]. This hierarchy can also be embedded into the framework of an action, leading to an extension of the third column in Table 7.1. More precisely, the de-forms ($p = D - 1$ in $D$ dimensions) and top-forms ($p = D$) appear in the action as Lagrange multipliers for the constraints on the embedding tensor.

Although the full hierarchy has an intricate gauge structure with interesting properties, in this text we will only consider its truncation to the $p \leq 2$-forms. More precisely, we study the gauge algebra structure of the $D = 4$ embedding tensor formalism up to 1- and 2-forms. We obtain the following results:

1. In order for the algebra to be closed on the 2-forms, one needs to add extra gauge transformations such that the total variation of the 2-forms can be written schematically as

$$\delta B = D\Xi + \ldots - \Theta\Phi \,, \tag{7.2}$$

   where we have introduced new local parameters $\Phi_{\mu\nu}{}^{MNP}$ whose properties will be explained in §7.3, and the dots denote extra terms that depend on the fields $\{A_\mu{}^M, B_{\mu\nu}{}^{MN}\}$ and parameters $\{\Lambda^M, \Xi_\mu{}^{MN}\}$ – recall equation (4.106) for their correct form. In the presence of an action, $\delta B$ also contains an extra non-vanishing term $\Delta B$.

   Given the new transformations in (7.2), we will show that the algebra has a closed form. On the other hand, once a particular form for the action is introduced and $\Delta B$ is included, the algebra of the modified gauge transformations is open. This means that the commutator of two modified gauge transformations only closes up to terms that are proportional to the equations of motion.

2. In all cases the algebra is reducible, which means that the 3 types of gauge transformations are not independent. This dependence is characterized by the so-called zero modes of the theory. In general, the system even turns out to be higher stage reducible; there exist zero modes for the zero modes etc. It is not clear if these higher stage zero modes stop after a finite number of steps, i.e., whether the algebra is finitely reducible.

3. In all cases the gauge algebra has a soft form, i.e., the structure constants are not really constant, but they depend on the fields in the theory.

Hence, we find that the algebra of symmetries has a structure that is very involved. Moreover, it has exactly the same properties for which the BV formalism was constructed. This brings us back to the main purpose of this paper, which is to provide a more concise framework for these complicated properties through a new formulation that connects the embedding tensor formalism to the BV formalism. In §7.4 we will first introduce the main ingredients of the BV formalism and then apply this structure to the embedding tensor formalism. The ultimate task is to find an expression for the extended action $S$ which is an expansion in the antifields and satisfies the master equation $(S, S) = 0$. We find that, at "zeroth order" in the antifields, the extended action is equal to the classical action. If, as mentioned above, we consider the gauge algebra in the absence of a classical action then the extended action has no zeroth order part. All higher order terms in $S$ depend on tensors that reflect the gauge structure of the vector and tensor fields. For example, at first order we find the gauge generators, second order contains the structure functions, etc. We will restrict our analysis to second order since this is enough to incorporate the most important properties of the algebra. Nevertheless, we expect that this investigation can be continued to arbitrary high orders in the antifields.

To summarize, we have motivated that a lot of the structure relations that appear in the different papers on the embedding formalism get unified in the master equation of the BV extended action. This result can be helpful in the future to gain more insight into the gauge structure at higher orders in the antifields. Moreover, we have now all the tools available to initiate the quantization of generic gauged field theories.

## 7.2 Notations and conventions

### The independent antisymmetric tensors

According to their upper index structure, the original antisymmetric 2-forms $B_{\mu\nu}{}^{NP}$ transform in the symmetric product of two vector representations under the rigid symmetry group. However, in §6.3 and also in the literature [61,137], one often introduces a different base of 2-forms, namely $\{B_{\mu\nu\alpha}\}$, that carry an adjoint index $\alpha$. The purpose of this section is to better explain the origin of these choices and to put them into a broader perspective.

We start from the observation that in all the relevant formulae of the original formalism in §4.3, the 2-forms $B_{\mu\nu}{}^{NP}$ are always contracted with a symmetric tensor $Y^M{}_{NP}$. Recall that the latter is defined by the symmetric combination of two symplectic matrices $X_{NP}{}^M$, i.e.,

$$Y^M{}_{NP} \equiv \frac{1}{2}\left(X_{NP}{}^M + X_{PN}{}^M\right) = \Theta_{(N}{}^\alpha (t_\alpha)_{P)}{}^M\,. \tag{7.3}$$

Since $Y^M{}_{NP}$ depends on the embedding tensor, its form is determined by the particular solution of the quadratic and linear constraints on $\Theta_N{}^\alpha$. In particular, it may happen that $Y^M{}_{NP}$ has less components than all symmetric $(NP)$ combinations. This can be quantified as follows. We introduce a projection operator $\mathbb{P}^{RS}{}_{NP}$ that is symmetric in $(RS)$ and $(NP)$ and that leaves $Y^M{}_{NP}$ invariant,

$$Y^M{}_{RS}\mathbb{P}^{RS}{}_{NP} = Y^M{}_{NP}\,. \tag{7.4}$$

In principle, the projector could be chosen as trivial: $\mathbb{P}^{RS}{}_{NP} = \delta_{(N}{}^{R}\delta_{P)}{}^{S}$. However, if the embedding tensor satisfies the locality and representation constraints, $\mathbb{P}^{RS}{}_{NP}$ can also be of lower rank such that it projects only onto the $(NP)$ that remain in $Y^{M}{}_{NP}$. We will show how this works in an explicit example in §7.2.

Since the antisymmetric tensors are contracted by $Y^{M}{}_{RS}\mathbb{P}^{RS}{}_{NP}$, the only tensors that survive this contraction are the ones that do not vanish upon multiplication with the projector $\mathbb{P}^{RS}{}_{NP}$. In the following, we will use the notation $B_{\mu\nu}{}^{\lceil RS \rfloor}$ to denote these non-vanishing 2-forms:

$$B_{\mu\nu}{}^{\lceil RS \rfloor} \equiv \mathbb{P}^{RS}{}_{NP} B_{\mu\nu}{}^{NP}\,, \qquad \text{and} \qquad Y^{M}{}_{NP} B_{\mu\nu}{}^{NP} = Y^{M}{}_{NP} B_{\mu\nu}{}^{\lceil NP \rfloor}\,. \tag{7.5}$$

This clarifies the origin of the different sets of 2-forms. Each set corresponds to a different choice of the projection operator. We give two examples.

- If $\mathbb{P}^{RS}{}_{NP} = \delta_{(N}{}^{R}\delta_{P)}{}^{S}$, the 2-forms that survive correspond to the original set $\{B_{\mu\nu}{}^{RS}\}$. For this choice, $\mathbb{P}^{RS}{}_{NP}$ has maximal rank.
- If the linear constraint is satisfied, we have shown in (6.34) that in the absence of anomalies, $Y^{M}{}_{NP}$ can be written as

  $$Y^{M}{}_{NP} = Z^{M\alpha}\Delta_{\alpha NP}\,. \tag{7.6}$$

  Therefore, we can identify $\Delta_{\alpha NP}$ with the non-vanishing part of a projector $\mathbb{P}^{RS}{}_{NP}$ and the corresponding 2-forms are $B_{\mu\nu\alpha}$.[4] This configuration corresponds to a choice of $\mathbb{P}^{RS}{}_{NP}$ with minimal rank. For more information about these statements in the context of an explicit example, we refer to §7.2.

For completeness, we note that exactly the same reasoning applies to the parameters $\Xi_{\mu}{}^{MN}$, which can be restricted to $\Xi_{\mu}{}^{\lceil MN \rfloor}$. Then the gauge transformations of the 2-forms $B_{\mu\nu}{}^{\lceil NP \rfloor}$ become

$$\delta(\Lambda,\Xi)B_{\mu\nu}{}^{\lceil NP \rfloor} = 2D_{[\mu}\Xi_{\nu]}{}^{\lceil NP \rfloor} + 2A_{[\mu}{}^{\lceil N}\delta A_{\nu]}{}^{P \rfloor} - 2\Lambda^{\lceil N}\mathcal{H}_{\mu\nu}{}^{P \rfloor} + \Delta B_{\mu\nu}{}^{\lceil NP \rfloor}\,, \tag{7.7}$$

where we introduced the general notation

$$A^{\lceil M}B^{N \rfloor} \equiv \mathbb{P}^{MN}{}_{RS} A^{R} B^{S}\,, \tag{7.8}$$

for some tensors $A^{R}$ and $B^{S}$, and $\Delta B_{\mu\nu}{}^{\lceil NP \rfloor}$ is either zero (in the absence of an action) or given by

$$\Delta B_{\mu\nu}{}^{\lceil NP \rfloor} = -2\Lambda^{\lceil N}\left(\mathcal{G}_{\mu\nu}{}^{P \rfloor} - \mathcal{H}_{\mu\nu}{}^{P \rfloor}\right)\,. \tag{7.9}$$

[4] In fact there is a subtlety related to the possible presence of vanishing symplectic matrices $(t_{\alpha})_{M}{}^{N}$ for certain values of $\alpha$. For these $\alpha$ the corresponding object $\Delta_{\alpha NP}$ also vanishes and there will be no 2-form with this index in the set $\{B_{\mu\nu\alpha}\}$.

In the remainder of this chapter, we will work within the framework of §4.3, but we make the replacements

$$B_{\mu\nu}{}^{MN} \to B_{\mu\nu}{}^{\lceil MN \rfloor}, \qquad \Xi_{\mu}{}^{MN} \to \Xi_{\mu}{}^{\lceil MN \rfloor} \tag{7.10}$$

everywhere. We will not specify the precise form of $\mathbb{P}^{MN}{}_{RS}$, but only use the property that it is symmetric in $(MN)$ and $(RS)$.

In order to further illustrate the meaning of the special brackets $\lceil .. \rfloor$, let us now consider a simple example. It is not completely general in the sense that we will only impose the representation constraint on the embedding tensor, but nevertheless it clarifies already a lot of the structure above.

### Example

We want to study the possible gaugings of $D = 4$, $\mathcal{N} = 2$ supergravity coupled to 1 vector multiplet (1 vector, two scalars), using the embedding tensor formalism.

The theory has a $G_{\text{global}} = SL(2,\mathbb{R})$ global symmetry group, we will denote its generators by $\delta_\alpha,\ \alpha = 1, 2, 3$. It leaves the scalar part of the Lagrangian invariant and works on the field strengths (and its duals) via an embedding in $Sp(4,\mathbb{R})$:

$$\begin{aligned} &\delta_\alpha F_{\mu\nu}{}^{M} = -F_{\mu\nu}{}^{N} (t_\alpha)_N{}^M \,, \quad M = 1, \ldots, 4\,, \\ &t_{\alpha[M}{}^{P} \Omega_{N]P} = 0\,. \end{aligned} \tag{7.11}$$

The index $M$ labels the vector from the vector multiplet, the graviphoton and their magnetic duals. The symplectic matrices $(t_\alpha)_M{}^P$ are

$$t_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \ t_2 = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \end{pmatrix}, \ t_3 = \begin{pmatrix} 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7.12}$$

**Embedding tensor.** We want to gauge a subgroup $G_0 \subset G_{\text{global}}$ by promoting some of the $\delta_\alpha$ to local transformations. The possible subgroups are selected by the embedding tensor $\Theta_M{}^\alpha$. From its index structure, we see that as a spurious object, it transforms in the $\mathbf{3} \times \mathbf{4} = \mathbf{2} + \mathbf{4} + \mathbf{6}$ of $SL(2,\mathbb{R})$. In Young Tableaux we have

$$\yng(2) \otimes \yng(3) = \yng(3,2) + \yng(4,1) + \yng(5)\,. \tag{7.13}$$

Of course, in order to define a consistent (and supersymmetric) gauging, constraints have to be imposed on the embedding tensor. The linear constraint will restrict the representation of $\Theta_M{}^\alpha$ to be the $\mathbf{2}$, as we will see in the next paragraph.

**Constraints.** In order to have a consistent and supersymmetric gauging, we need to impose the constraints $D_{MNP} = 0$ and $Q_{MN}{}^{\gamma} = 0$ on $\Theta_M{}^{\alpha}$. For the sake of clarity, we will only impose the representation constraint which tells us that $\Theta_M{}^{\alpha}$ should transform in the **2**. The latter takes the following form:

$$\Theta_M{}^{\alpha} \quad = \quad \begin{pmatrix} a^1 & 0 & -a^2 \\ 0 & \sqrt{3}a^2 & 0 \\ a^2 & a^1 & 0 \\ 0 & 0 & -\sqrt{3}a^1 \end{pmatrix} \equiv a^i (\Theta_i)_M{}^{\alpha} \,, \qquad a^1, a^2 \in \mathbb{R} \,, \quad (7.14)$$

which defines the matrices $(\Theta_i)_M{}^{\alpha}$. A way to see that this particular $\Theta_M{}^{\alpha}$ transforms in the **2** is to check that

$$\delta_\beta \Theta_M{}^{\alpha} = (t_\beta)_M{}^N \Theta_N{}^{\alpha} + f_{\beta\gamma}{}^{\alpha} \Theta_M{}^{\gamma} = a^i (t_\beta)_i{}^j (\Theta_j)_M{}^{\alpha} \,, \qquad (7.15)$$

where $f_{\beta\gamma}{}^{\alpha}$ are the structure constants of $G_{\text{global}} = SL(2,\mathbb{R})$ and the matrices $(t_\alpha)_i{}^j$ are the generators $\delta_\alpha$ in the fundamental representation:

$$t_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad t_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \qquad t_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \qquad (7.16)$$

**Zero modes.** Now that we have defined an embedding tensor that satisfies the linear constraint and since the symplectic embedding for the $\delta_\alpha$ is known, we can construct the objects $Y^P{}_{MN}$. The index $P$ runs over 4 values and the symmetric combination $(MN)$ over 10 values. So $Y^P{}_{MN}$ can be seen as an $(4 \times 10)$-matrix, where we adopt the following convention for the order of the columns:

| $MN$ | 11 | 12 | 13 | 14 | 22 | 23 | 24 | 33 | 34 | 44 |
|---|---|---|---|---|---|---|---|---|---|---|
| $(MN)$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

(7.17)

Since $\Theta_M{}^{\alpha}$ satisfies the linear constraint, the expression for $Y^P{}_{MN}$ can be found in two ways, either via a direct calculation using (7.3), or via the expression in (7.6). As expected, both approaches give the same result, namely

$$Y^P{}_{MN} = \begin{pmatrix} a^1 & 0 & -\frac{a^2}{2} & 0 & 0 & 0 & \frac{3a^2}{2} & 0 & -\frac{\sqrt{3}a^1}{2} & 0 \\ 0 & -\frac{3a^1}{2} & 0 & 0 & 0 & 0 & 0 & \sqrt{3}a^1 & 0 & 0 \\ 0 & \frac{\sqrt{3}a^2}{2} & \frac{a^1}{2} & 0 & 0 & 0 & -\frac{3a^1}{2} & -a^2 & 0 & 0 \\ -\sqrt{3}a^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3a^2}{2} & 0 \end{pmatrix} . \qquad (7.18)$$

Since we will need them later, let us also write down the components of $\Delta_{\alpha MN}$, which form a $(3 \times 10)$-matrix:

$$\Delta_{\alpha MN} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 \\ 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \end{pmatrix} . \qquad (7.19)$$

**Projection operators and bases of 2-forms.** In the last step, we will determine the constant projection operators $\mathbb{P}^{MN}{}_{RS}$ that leave the $Y^P{}_{MN}$ invariant, i.e. we will solve equation (7.4) together with

$$\mathbb{P}^{MN}{}_{PQ}\mathbb{P}^{PQ}{}_{RS} = \mathbb{P}^{MN}{}_{RS} \, . \tag{7.20}$$

Several solutions are possible, but the *minimal-rank* $(10 \times 10)$-matrix is the following:

$$\mathbb{P}^{MN}{}_{RS} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ & & & & & & & & & \\ & & & & & 0_{7\times 10} & & & & \\ & & & & & & & & & \end{pmatrix} . \tag{7.21}$$

This projector defines a minimal set of 2-forms $B_{\mu\nu}{}^{\lceil MN \rfloor}$ via the definition (7.5),

$$\begin{aligned} B_{\mu\nu}{}^{\lceil 11 \rfloor} &= B_{\mu\nu}{}^{11} - \frac{\sqrt{3}}{2} B_{\mu\nu}{}^{34} \, , \\ B_{\mu\nu}{}^{\lceil 12 \rfloor} &= B_{\mu\nu}{}^{12} - \frac{2}{\sqrt{3}} B_{\mu\nu}{}^{33} \, , \\ B_{\mu\nu}{}^{\lceil 13 \rfloor} &= B_{\mu\nu}{}^{13} - 3 B_{\mu\nu}{}^{24} \, , \\ B_{\mu\nu}{}^{\lceil 14 \rfloor} &= B_{\mu\nu}{}^{\lceil 22 \rfloor} = \ldots = 0 \, . \end{aligned} \tag{7.22}$$

On the other hand, we can also define $B_{\mu\nu\alpha} \equiv \Delta_{\alpha MN} B_{\mu\nu}{}^{MN}$. Applied to our example, this gives

$$\begin{aligned} B_{\mu\nu 1} &= -B_{\mu\nu}{}^{13} + 3 B_{\mu\nu}{}^{24} \, , \\ B_{\mu\nu 2} &= 2 B_{\mu\nu}{}^{11} - \sqrt{3} B_{\mu\nu}{}^{34} \, , \\ B_{\mu\nu 3} &= \sqrt{3} B_{\mu\nu}{}^{12} - 2 B_{\mu\nu}{}^{33} \, . \end{aligned} \tag{7.23}$$

Comparing (7.22) to (7.23), we see that we can identify

$$B_{\mu\nu 1} = -B_{\mu\nu}{}^{\lceil 13 \rfloor}, \quad B_{\mu\nu 2} = 2 B_{\mu\nu}{}^{\lceil 11 \rfloor}, \quad B_{\mu\nu 3} = \sqrt{3} B_{\mu\nu}{}^{\lceil 12 \rfloor} \, . \tag{7.24}$$

Therefore, we conclude that $\{B_{\mu\nu}{}^{\lceil MN \rfloor}\}$ and $\{B_{\mu\nu\alpha}\}$ really are the same sets of fields, up to some constants. This proves our conjectured form of $\mathbb{P}$ and the corresponding 2-forms below equation (7.6). In particular it is clear from this example that $\Delta_{\alpha MN}$ is a submatrix of the full projection operator $\mathbb{P}^{MN}{}_{RS}$.

## DeWitt notation and gauge generators

We have now identified the independent 2-forms and their gauge transformations. Together with the other ingredients in Table 7.1 they make up the embedding tensor formalism. Before we go on to investigate its gauge structure, let us still present some new notations that will simplify the discussion later on. More precisely, we will introduce the DeWitt notation that provides a compact and transparent way of writing down general field theories. The different fields are denoted by $\phi^i$, where the index $i = 1, \ldots, n$ can label space-time indices $\mu$, $\nu$, ... for tensor fields, spinor indices for fermion fields, and/or an index distinguishing different types of generic fields. The fields are also functions of space-time, and we will adopt the convention that the appearance of a discrete DeWitt index also indicates the presence of a space-time variable. We then use a generalized summation convention in which a repeated discrete index implies not only a sum over that index but also an integration over the corresponding space-time variable.

In our case, the general notation $\phi^i$ is an abbreviation for the collection of bosonic tensor fields,

$$\phi^i \in \left\{ A_\mu{}^M(x),\, B_{\mu\nu}{}^{\lceil MN \rfloor}(x) \right\} . \tag{7.25}$$

It means that the $i$-index takes the following discrete values: $\{\mu M,\, \mu\nu[\![MN]\!]\}$, where one should remember that the space-time indices and (combinations of) vector indices are not at the same level.

Furthermore, we have a set of $m_0$ non-trivial bosonic gauge transformations. In the DeWitt notation, they take the following form [5]

$$\delta\phi^i = \boldsymbol{R}^i{}_{a_0}(\phi)\varepsilon^{a_0} , \qquad \text{with } a_0 = 1,\, 2,\, \ldots,\, m_0 . \tag{7.27}$$

The infinitesimal gauge parameters $\varepsilon^{a_0}$ are arbitrary functions of the space-time variable $x$, and $\boldsymbol{R}^i{}_{a_0}$ denotes the generators of the gauge transformations. The different types of gauge parameters that we have introduced so far are:

$$\varepsilon^{a_0} \in \left\{ \Lambda^M(x),\, \Xi_\mu{}^{\lceil MN \rfloor}(x) \right\} , \tag{7.28}$$

thus, $a_0 \in \{M,\, \mu[\![MN]\!]\}$. The gauge generators $\boldsymbol{R}^i{}_{a_0}$ in the embedding tensor formalism can be computed by comparing the field transformations to the general expression (7.27). For the vector fields, we read off from (4.93) that their

[5] Without the use of a compact summation convention, this relation would be represented as

$$\delta\phi^i(x) = \int \mathrm{d}y\, \boldsymbol{R}^i{}_{a_0}(x,y)\, \varepsilon^{a_0}(y) . \tag{7.26}$$

transformations are generated by [6]

$$\begin{aligned} \boldsymbol{R}_{\mu}{}^{M}{}_{K} &= D_{\mu}{}^{M}{}_{K} \,, & (7.30) \\ \boldsymbol{R}_{\mu}{}^{M\,\nu}{}_{\lceil KL \rfloor} &= -\delta^{\nu}_{\mu} Y^{M}{}_{KL} \,. & (7.31) \end{aligned}$$

The $\mu M$-indices in these equations correspond to the $i$-index in $\boldsymbol{R}^{i}{}_{a_0}$. The $a_0$-index takes the values $K$ in (7.30) and $\nu\llbracket KL \rrbracket$ in (7.31). Let us for once give a more detailed discussion on how (7.30) and (7.31) are obtained (all the subsequent results can be found in a similar way). We choose $\phi^i$ to be equal to a vector, then (7.27) can be written as

$$\begin{aligned} \delta A_{\mu}{}^{M}(x) &= \int \mathrm{d}^4 y\, \boldsymbol{R}_{\mu}{}^{M}{}_{a_0}(x,y)\varepsilon^{a_0}(y) & (7.32) \\ &= \int \mathrm{d}^4 y \left[ \boldsymbol{R}_{\mu}{}^{M}{}_{K}(x,y)\Lambda^{K}(y) + \boldsymbol{R}_{\mu}{}^{M\,\nu}{}_{\lceil KL \rfloor}(x,y)\Xi_{\nu}{}^{\lceil KL \rfloor}(y) \right] \end{aligned}$$

In the first line, we explicitly wrote down the integral over the space-time variable $y$, which was hidden in the summation over $a_0$. In the second line, we further worked out the summation over $a_0$. This result should now be compared to (4.93), and we find

$$\begin{aligned} \boldsymbol{R}_{\mu}{}^{M}{}_{K}(x,y) &= \left( \delta_{K}{}^{M} \frac{\partial}{\partial x^{\mu}} - A_{\mu}{}^{Q}(x) X_{QK}{}^{M} \right) \delta(x-y) \\ &= D_{\mu}{}^{M}{}_{K}(x)\delta(x-y) \,, & (7.33) \\ \boldsymbol{R}_{\mu}{}^{M\,\nu}{}_{\lceil KL \rfloor}(x,y) &= -\delta^{\nu}_{\mu} Y^{M}{}_{KL}\delta(x-y) \,. & (7.34) \end{aligned}$$

In the following, we will suppress the space-time variables and delta functions, such that (7.33) and (7.34) reduce to the expressions (7.30) and (7.31) respectively.

[6] In the following, we will use the notation $D_{\mu}{}^{N_1 \ldots N_p}{}_{M_1 \ldots M_p}$, which is a particular derivation operation. For $p = 1$, it is defined as follows:

$$D_{\mu}{}^{N}{}_{M} T^{M} \equiv \left( \delta_{M}{}^{N} \partial_{\mu} + A_{\mu}{}^{Q} X_{QM}{}^{N} \right) T^{M} = D_{\mu} T^{N} \,, \qquad (7.29)$$

for some object $T^M$ that transforms with a vector index. For $p > 1$ a similar definition holds, but its precise form will not be important for us. We refer to appendix A of [3] for more details.

Likewise, the tensors that generate the transformations of the 2-forms $B_{\mu\nu}{}^{\lceil NP\rfloor}$ can be determined from (7.7):

$$\left\{\begin{array}{rcl} \boldsymbol{R}_{\mu\nu}{}^{\lceil NP\rfloor}{}_K &=& 2A_{[\mu}{}^{\lceil N}D_{\nu]}{}^{P\rfloor}{}_K - 2\delta_K{}^{\lceil N}\,\mathcal{H}_{\mu\nu}{}^{P\rfloor} \\ && \text{for}\ \ \Delta B_{\mu\nu}{}^{\lceil NP\rfloor}=0\,, \\ \widetilde{\boldsymbol{R}}_{\mu\nu}{}^{\lceil NP\rfloor}{}_K &=& 2A_{[\mu}{}^{\lceil N}D_{\nu]}{}^{P\rfloor}{}_K - 2\delta_K{}^{\lceil N}\,\mathcal{G}_{\mu\nu}{}^{P\rfloor} \\ && \text{for}\ \ \Delta B_{\mu\nu}{}^{\lceil NP\rfloor} = -2\Lambda^{\lceil N}\left(\mathcal{G}-\mathcal{H}\right)_{\mu\nu}{}^{P\rfloor}, \end{array}\right. \tag{7.35}$$

$$\boldsymbol{R}_{\mu\nu}{}^{\lceil NP\rfloor\rho}{}_{\lceil RS\rfloor} \;=\; 2D_{[\mu}{}^{NP}{}_{RS}\delta^{\rho}_{\nu]} - 2A_{[\mu}{}^{\lceil N}Y^{P\rfloor}{}_{RS}\delta^{\rho}_{\nu]}\,.$$

Behind the braces, we made a distinction between the gauge transformations without the specification of an action (i.e. $\Delta B=0$), and the gauge transformations that leave the action (4.101) invariant. In order to tell the difference between these two cases, we have added a tilde to the generators on the third line. In the following, we will use the more general notation $\widetilde{\boldsymbol{R}}^i{}_{a_0}$ to indicate all generators that leave the action (4.101) invariant. It is clear that $\widetilde{\boldsymbol{R}}^i{}_{a_0} = \boldsymbol{R}^i{}_{a_0}$, except for $i=\mu\nu\llbracket NP\rrbracket$, $a_0=K$.

Given the precise form of the gauge generators, the next step is compute the detailed properties of the gauge algebra structure. This will be the subject of the next section.

## 7.3 Structure of the gauge algebra

In this section we provide an answer to the following questions:

1. Is the algebra closed? For a closed algebra, the commutator of two gauge transformations leads again to a linear combination of transformations, with new parameters that depend on the fields and the original parameters. Part of this question was already answered in §4.3, where we checked that the algebra indeed closes on the vectors with transformations $\delta(\Lambda)$ and $\delta(\Xi)$. Here we will extend this result to the 2-forms.

2. Are the structure constants really ‘constant’, or are they functions of the fields?

3. Is the gauge algebra (ir)reducible? This question addresses the (in)dependence of the different gauge transformations, which is important if we want to determine the independent degrees of freedom in the theory.

These issues will be dealt with in several steps. In §7.3 we finish our discussion on the gauge algebra commutators and show that the algebra closes for $\Delta B=0$ and is generally open for $\Delta B \neq 0$. Once all the commutators are known, the structure 'constants' can easily be determined. They turn out to be field dependent and thus we have a soft algebra. Finally, in §7.3 and §7.3 the dependencies among the gauge transformations will be investigated. We show that the algebra is higher stage reducible and we provide an explicit construction of the zero modes.

### Closure of the algebra

We make a clear distinction between the generators $\boldsymbol{R}^i{}_{a_0}$ that are part of the embedding tensor formalism without the specification of an action, and the generators $\widetilde{\boldsymbol{R}}^i{}_{a_0}$ that appear in the Lagrangian description. The difference between these two cases, which is captured by $\Delta B$, leads to distinct conclusions about the properties of the corresponding gauge algebra. In the first part, we consider the formalism without an action.

#### Formalism without an action

In order to have a closed algebra, the gauge transformations need to satisfy the following relation:

$$[\delta_1, \delta_2]\phi^i = \boldsymbol{R}^i{}_{a_0} \boldsymbol{T}^{a_0}{}_{b_0 c_0}\, \varepsilon_1^{b_0} \varepsilon_2^{c_0}\,. \tag{7.36}$$

The $\boldsymbol{T}^{a_0}{}_{b_0 c_0}$ are antisymmetric tensors under the interchange of indices $b_0$ and $c_0$. They are called the 'structure constants' of the algebra, although in general, they depend on the fields of the theory.

From (4.95) we know that the commutator of two $\delta(\Lambda)$ transformations on the gauge fields leads again to a linear combination of a $\delta(\Lambda)$ and a $\delta(\Xi)$ transformation. Likewise, one can show that

$$[\delta(\Lambda), \delta(\Xi)] A_\mu{}^M = 0\,, \tag{7.37}$$

$$[\delta(\Xi_1), \delta(\Xi_2)] A_\mu{}^M = 0\,. \tag{7.38}$$

We conclude that the gauge algebra with transformations $\delta(\Lambda)$ and $\delta(\Xi)$ indeed satisfies the relation (7.36) on the vector fields. The only non-vanishing structure constants are

$$\boldsymbol{T}^M{}_{RS} = X_{[RS]}{}^M\,, \qquad \boldsymbol{T}_\mu{}^{\lceil MN \rfloor}{}_{RS} = \delta_R{}^{\lceil M} D_\mu{}^{N \rfloor}{}_S - \delta_S{}^{\lceil M} D_\mu{}^{N \rfloor}{}_R\,, \tag{7.39}$$

where both $\boldsymbol{T}$'s are antisymmetric in $[RS]$.

On the other hand, we have not yet checked whether the algebra also closes on the 2-forms. Let us therefore compute the non-vanishing commutators. We find that:

$$\begin{aligned}
[\delta(\Lambda_1),\delta(\Lambda_2)]B_{\mu\nu}{}^{\lceil NP\rfloor} &= \delta(\Lambda_3)B_{\mu\nu}{}^{\lceil NP\rfloor}+\delta(\Xi_3)B_{\mu\nu}{}^{\lceil NP\rfloor}\\
&\quad -Y^{NP}{}_{M\lceil RS\rfloor}\left(\Lambda_1^M\mathcal{H}_{\mu\nu}{}^{\lceil R}\Lambda_2^{S\rfloor}-(1\leftrightarrow 2)\right),\\
[\delta(\Lambda),\delta(\Xi)]B_{\mu\nu}{}^{\lceil NP\rfloor} &= -Y^{NP}{}_{M\lceil RS\rfloor}\left(-2\Xi_{[\mu}{}^{\lceil RS\rfloor}D_{\nu]}\Lambda^M\right),\\
[\delta(\Xi_1),\delta(\Xi_2)]B_{\mu\nu}{}^{\lceil NP\rfloor} &= Y^{NP}{}_{M\lceil RS\rfloor}Y^M{}_{QT}\left(\Xi_{1[\mu}{}^{\lceil QT\rfloor}\Xi_{2\nu]}{}^{\lceil RS\rfloor}-(1\leftrightarrow 2)\right),
\end{aligned} \tag{7.40}$$

with

$$Y^{NP}{}_{M\lceil RS\rfloor}\equiv 2\left(\delta_M{}^{\lceil N}Y^{P\rfloor}{}_{RS}-X_{M\lceil R}{}^{\lceil N}\delta_{S\rfloor}{}^{P\rfloor}\right). \tag{7.41}$$

We note that the contraction of this tensor with $Y^Q{}_{NP}$ vanishes,

$$Y^Q{}_{NP}Y^{NP}{}_{M\lceil RS\rfloor}=0\,, \tag{7.42}$$

which is a relation that will be important later on.

Let us now study the commutation relations (7.40) in more detail. Clearly, the closure condition (7.36) is not satisfied since each of the commutators contains an extra term that is proportional to $Y^{NP}{}_{M\lceil RS\rfloor}$. There is however a way to restore closure of the algebra, which is completely analogous to our treatment in §4.3 for the 1-forms: we extend the original gauge transformations with a new local transformation that is proportional to $Y^{NP}{}_{M\lceil RS\rfloor}$,

$$\delta(\Lambda,\Xi)B_{\mu\nu}{}^{\lceil MN\rfloor}\quad\rightarrow\quad\delta(\Lambda,\Xi,\Phi)B_{\mu\nu}{}^{\lceil MN\rfloor}=\delta(\Lambda,\Xi)B_{\mu\nu}{}^{\lceil MN\rfloor}+\delta(\Phi)B_{\mu\nu}{}^{\lceil MN\rfloor}, \tag{7.43}$$

with [7]

$$\delta(\Phi)B_{\mu\nu}{}^{\lceil MN\rfloor}=-Y^{MN}{}_{P\lceil RS\rfloor}\Phi_{\mu\nu}{}^{\lceil P\lceil RS\rfloor\rfloor} \tag{7.44}$$

and new local parameters $\Phi_{\mu\nu}{}^{\lceil P\lceil RS\rfloor\rfloor}(x)$. The original set of gauge transformations in (7.28) should therefore be replaced by $\{\delta(\Lambda),\delta(\Xi),\delta(\Phi)\}$ with local parameters

$$\varepsilon^{a_0}\in\left\{\Lambda^M(x),\,\Xi_\mu{}^{\lceil MN\rfloor}(x),\,\Phi_{\mu\nu}{}^{\lceil M\lceil NP\rfloor\rfloor}(x)\right\}. \tag{7.45}$$

[7] We have used a special notation here with nested brackets $\lceil.\lceil.\rfloor\rfloor$. One can think of it as a generalization of $\lceil.\rfloor$ in the sense that we define a projector $\mathbb{P}^{K\lceil MN\rfloor}{}_{P\lceil RS\rfloor}$ that leaves $Y^{NQ}{}_{K\lceil LM\rfloor}$ invariant and we replace the indices in each 3-index object that is fully contracted with $\mathbb{P}^{K\lceil MN\rfloor}{}_{P\lceil RS\rfloor}$ by $\lceil K\lceil LM\rfloor\rfloor$. More information can be found in appendix A of [3].

It is clear that the index $a_0$ takes an extra value $\mu\nu\llbracket M\lceil NP\rfloor\rrbracket$ and the corresponding gauge generators are

$$\boldsymbol{R}_\rho{}^{K\,\mu\nu}{}_{\lceil M\lceil NP\rfloor\rfloor} = 0\,, \tag{7.46}$$

$$\boldsymbol{R}_{\rho\sigma}{}^{\lceil KL\rfloor\,\mu\nu}{}_{\lceil M\lceil NP\rfloor\rfloor} = -\delta^{[\mu\nu]}_{\rho\sigma}Y^{KL}{}_{M\lceil NP\rfloor}\,. \tag{7.47}$$

With this new set of gauge transformations, it is easy to check that the algebra closes:

$$\begin{aligned}
[\delta(\Lambda_1),\delta(\Lambda_2)]B_{\mu\nu}{}^{\lceil NP\rfloor} &= \delta(\Lambda_3)B_{\mu\nu}{}^{\lceil NP\rfloor} + \delta(\Xi_3)B_{\mu\nu}{}^{\lceil NP\rfloor} + \delta(\Phi_3)B_{\mu\nu}{}^{\lceil NP\rfloor}\,,\\
[\delta(\Lambda),\delta(\Xi)]B_{\mu\nu}{}^{\lceil NP\rfloor} &= \delta(\Phi_4)B_{\mu\nu}{}^{\lceil NP\rfloor}\,,\\
[\delta(\Xi_1),\delta(\Xi_2)]B_{\mu\nu}{}^{\lceil NP\rfloor} &= \delta(\Phi_5)B_{\mu\nu}{}^{\lceil NP\rfloor}\,.
\end{aligned} \tag{7.48}$$

The commutators satisfy the relation (7.36) and the parameters $\Phi_3$, $\Phi_4$ and $\Phi_5$ determine the precise form of the structure constants. From (7.48) one sees that:

$$\Phi_{3\,\mu\nu}{}^{\lceil M\lceil RS\rfloor\rfloor} = \Lambda_1^{\lceil M}\mathcal{H}_{\mu\nu}{}^{\lceil R}\Lambda_2^{S\rfloor\rfloor} - (1\leftrightarrow 2)\,, \tag{7.49}$$

$$\Phi_{4\,\mu\nu}{}^{\lceil M\lceil RS\rfloor\rfloor} = -2\Xi_{[\mu}{}^{\lceil\lceil RS\rfloor}D_{\nu]}\Lambda^{M\rfloor}\,, \tag{7.50}$$

$$\Phi_{5\,\mu\nu}{}^{\lceil M\lceil RS\rfloor\rfloor} = Y^{\lceil M}{}_{QT}\left(\Xi_{1[\mu}{}^{\lceil RS\rfloor\rfloor}\Xi_{2\nu]}{}^{QT} - (1\leftrightarrow 2)\right)\,, \tag{7.51}$$

which leads to an expression for the remaining non-vanishing structure constants:

$$\begin{aligned}
\boldsymbol{T}_{\mu\nu}{}^{\lceil M\lceil NP\rfloor\rfloor}{}_{RS} &= \delta_R{}^{\lceil M}\mathcal{H}_{\mu\nu}{}^{\lceil N}\delta_S{}^{P\rfloor\rfloor} - \delta_S{}^{\lceil M}\mathcal{H}_{\mu\nu}{}^{\lceil N}\delta_R{}^{P\rfloor\rfloor}\,,\\
\boldsymbol{T}_{\mu\nu}{}^{\lceil M\lceil NP\rfloor\rfloor}{}_R{}^\rho{}_{\lceil ST\rfloor} &= -\boldsymbol{T}_{\mu\nu}{}^{\lceil M\lceil NP\rfloor\rfloor\,\rho}{}_{\lceil ST\rfloor\,R} \;=\; -2\delta_S{}^{\lceil\lceil N}\delta_T{}^{P\rfloor}\delta^\rho_{[\mu}D_{\nu]}{}^{M\rfloor}{}_R\,,\\
\boldsymbol{T}_{\mu\nu}{}^{\lceil M\lceil NP\rfloor\rfloor\,\rho}{}_{\lceil QR\rfloor}{}^\sigma{}_{\lceil ST\rfloor} &= \delta^\rho_{[\mu}\delta^\sigma_{\nu]}\left(Y^{\lceil M}{}_{QR}\delta_S{}^{\lceil N}\delta_T{}^{P\rfloor\rfloor} + Y^{\lceil M}{}_{ST}\delta_Q{}^{\lceil N}\delta_R{}^{P\rfloor\rfloor}\right)\,.
\end{aligned} \tag{7.52}$$

We conclude this first part with the observation that the structure constants are not really constant, but they depend on the fields in the theory. This can be seen for example from (7.39) which depends on the vectors, and (7.52) which depends on both the vectors and 2-forms. Due to this field dependence of the structure functions, the gauge algebra is often called a 'soft algebra'.

### Formalism with an action

Our treatment in the previous section can now be generalized to the case where the embedding tensor formalism is incorporated into the framework of an action.

This has an effect on the gauge transformations, i.e. $\Delta B_{\mu\nu}{}^{\lceil MN \rfloor}$ takes the value in (4.107) and the generators $\boldsymbol{R}^i{}_{a_0}$ are replaced by $\widetilde{\boldsymbol{R}}^i{}_{a_0}$. So far, the latter have only been defined for the indices $i \in \{\mu M,\, \mu\nu [\![ MN ]\!]\}$ and $a_0 \in \{M, \mu [\![ MN ]\!]\}$. We will see in due course that the $a_0$ have to be extended as in (7.45), but for the time being, we only consider the smaller set. In §6.2 we saw that the transformations $\widetilde{\boldsymbol{R}}^i{}_{a_0}$ leave the action invariant, which is expressed by the Noether identities:

$$\partial_i S_0 \, \widetilde{\boldsymbol{R}}^i{}_{a_0} = 0 \,. \tag{7.53}$$

The most general solution to the Noether identities is a gauge transformation, up to terms proportional to the equations of motion:

$$\partial_i S_0 \, \lambda^i = 0 \quad \Leftrightarrow \quad \lambda^i = \widetilde{\boldsymbol{R}}^i{}_{a_0} \chi^{a_0} + \partial_j S_0 \, T^{ji} \,, \tag{7.54}$$

for some tensors $\chi^{a_0}$ and $T^{ij} = -T^{ji}$. The last term in (7.54) is known as a trivial gauge transformation, and it is easily checked that the action is invariant under these transformations due to the antisymmetry of $T^{ij}$ in $[ij]$. A particular choice for $\lambda^i$ in (7.54) would be the commutator of two gauge transformations on a field: $\lambda^i = [\delta_1, \delta_2]\phi^i$. Since for this particular choice, $\partial_i S_0 \, \lambda^i = 0$ is trivially satisfied due to (7.53), equation (7.54) tells us that $[\delta_1, \delta_2]\phi^i$ is of the form

$$[\delta_1, \delta_2]\phi^i = \widetilde{\boldsymbol{R}}^i{}_{a_0} \chi^{a_0} + \partial_j S_0 \, T^{ji} \,. \tag{7.55}$$

Since the left hand side is proportional to the antisymmetric combination of two gauge parameters, $\varepsilon_1^{[a_0} \varepsilon_2^{b_0]}$, so should be the right hand side. We can factor out these parameters and write

$$[\delta_1, \delta_2]\phi^i = \widetilde{\boldsymbol{R}}^i{}_{a_0} \widetilde{\boldsymbol{T}}^{a_0}{}_{b_0 c_0} \, \varepsilon_1^{b_0} \varepsilon_2^{c_0} + \partial_j S_0 \, \widetilde{\boldsymbol{E}}^{ij}{}_{a_0 b_0} \, \varepsilon_1^{a_0} \varepsilon_2^{b_0} \,. \tag{7.56}$$

This is the generalization of equation (7.36). The first term on the right hand side has a familiar form, with $\widetilde{\boldsymbol{T}}^{a_0}{}_{b_0 c_0}$ the structure 'constants' that are antisymmetric in $[b_0 c_0]$. The second term depends on the equations of motion multiplied by some $\widetilde{\boldsymbol{E}}$-tensors that are antisymmetric in both $[ij]$ and $[a_0 b_0]$. If these tensors do not vanish, the algebra only closes on-shell (i.e. when $\partial_j S_0 = 0$).

To summarize, the Noether identities impose a particular form for the gauge algebra, given by (7.56). We will now check whether (7.56) is indeed fulfilled for the 1- and 2-forms that arise in the embedding tensor formalism. Our results are as follows:

**Commutators on the 1-forms.** These do not change, i.e. (4.95), (7.37) and (7.38) are still valid. The reason is that the generators $\widetilde{\boldsymbol{R}}^i{}_{a_0}$ are equal to the generators $\boldsymbol{R}^i{}_{a_0}$ for $i = M$ and arbitrary $a_0$. This also means that the structure functions take the same values:

$$\widetilde{\boldsymbol{T}}^M{}_{RS} = \boldsymbol{T}^M{}_{RS} \,, \qquad \widetilde{\boldsymbol{T}}_\mu{}^{\lceil MN \rfloor}{}_{RS} = \boldsymbol{T}_\mu{}^{\lceil MN \rfloor}{}_{RS} \,. \tag{7.57}$$

The corresponding $\widetilde{\boldsymbol{E}}$-tensors in (7.56) all vanish.

**Commutators on the 2-forms.** These are slightly more involved. Let us start with the easiest case, which is the commutator of a $\delta(\Lambda)$ and a $\delta(\Xi)$ transformation. We find

$$[\delta(\Lambda), \delta(\Xi)] B_{\mu\nu}{}^{\lceil MN \rfloor} = -Y^{MN}{}_{Q\lceil RS \rfloor} \left( -2\Xi_{[\mu}{}^{\lceil RS \rfloor} D_{\nu]} \Lambda^Q \right) , \tag{7.58}$$

which is exactly the same expression as in (7.40). Consistency of the algebra requires the introduction of a new local transformation of the 2-forms, identical to (7.43). The corresponding generators are

$$\widetilde{\boldsymbol{R}}_\mu{}^{M\,\rho\sigma}{}_{\lceil Q \lceil RS \rfloor \rfloor} = 0 \,, \tag{7.59}$$

$$\widetilde{\boldsymbol{R}}_{\mu\nu}{}^{\lceil MN \rfloor\,\rho\sigma}{}_{\lceil Q \lceil RS \rfloor \rfloor} = -\delta^{[\rho\sigma]}_{\mu\nu} Y^{MN}{}_{Q\lceil RS \rfloor} \,, \tag{7.60}$$

and the index range $a_0$ has to be extended to

$$a_0 \in \{M, \mu\llbracket MN \rrbracket, \mu\nu\llbracket M \llbracket NP \rrbracket\rrbracket\} \,. \tag{7.61}$$

It is important to note that the form of the algebra in (7.56) should still be valid for this extended set of indices. Since (7.56) is a consequence of the Noether identities (7.53), it is enough to check that the latter also hold for $a_0 = \mu\nu\llbracket M \llbracket NP \rrbracket\rrbracket$. Indeed,

$$\begin{aligned} \partial_i S_0 \, \widetilde{\boldsymbol{R}}^{i\,\mu\nu}{}_{\lceil M \lceil NP \rfloor \rfloor} &= \frac{\partial S_0}{\partial A_\rho{}^R} \widetilde{\boldsymbol{R}}_\rho{}^{R\,\mu\nu}{}_{\lceil M \lceil NP \rfloor \rfloor} + \frac{\partial S_0}{\partial B_{\rho\sigma}{}^{\lceil RS \rfloor}} \widetilde{\boldsymbol{R}}_{\rho\sigma}{}^{\lceil RS \rfloor\,\mu\nu}{}_{\lceil M \lceil NP \rfloor \rfloor} \\ &\sim \left( Y^Q{}_{RS} \right) \left( -\delta^{[\mu\nu]}_{\rho\sigma} Y^{RS}{}_{M\lceil NP \rfloor} \right) = 0 \,. \end{aligned} \tag{7.62}$$

We used (7.59) and the fact that each 2-form $B_{\mu\nu}{}^{\lceil RS \rfloor}$ is contracted with a tensor $Y^Q{}_{RS}$ in the action. Then the last line vanishes because of the orthogonality of the $Y$-tensors, see (7.42).

Due to the introduction of the new transformations $\delta(\Phi)$, relation (7.58) can be written as

$$[\delta(\Lambda), \delta(\Xi)] B_{\mu\nu}{}^{\lceil MN \rfloor} = \delta(\widetilde{\Phi}_4) B_{\mu\nu}{}^{\lceil MN \rfloor} \,, \tag{7.63}$$

with $\widetilde{\Phi}_4 = \Phi_4$. If we compare this to the general expression (7.56), it is clear that all the corresponding $\widetilde{\boldsymbol{E}}$-tensors vanish and the non-zero structure function is identical to the one in (7.52):

$$\widetilde{\boldsymbol{T}}_{\mu\nu}{}^{\lceil M \lceil NP \rfloor \rfloor}{}_R{}^\rho{}_{\lceil ST \rfloor} = \boldsymbol{T}_{\mu\nu}{}^{\lceil M \lceil NP \rfloor \rfloor}{}_R{}^\rho{}_{\lceil ST \rfloor} \,. \tag{7.64}$$

Similar results hold for the commutator of two $\delta(\Xi)$ transformations on the 2-forms. We have

$$[\delta(\Xi_1), \delta(\Xi_2)] B_{\mu\nu}{}^{\lceil MN \rfloor} = \delta(\widetilde{\Phi}_5) B_{\mu\nu}{}^{\lceil MN \rfloor} \,, \tag{7.65}$$

with $\widetilde{\Phi}_5 = \Phi_5$. Also here the $\widetilde{\boldsymbol{E}}$-tensors vanish and $\widetilde{\boldsymbol{T}}_{\mu\nu}{}^{\lceil M\lceil NP\rfloor\rfloor\rho}{}_{\lceil QR\rfloor}{}^{\sigma}{}_{\lceil ST\rfloor}$ is given in (7.52).

Finally, we calculate the commutator of two $\delta(\Lambda)$ transformations:

$$
\begin{aligned}
&[\delta(\Lambda_1), \delta(\Lambda_2)] B_{\mu\nu}{}^{\lceil MN\rfloor} = \\
&\qquad \delta(\widetilde{\Lambda}_3) B_{\mu\nu}{}^{\lceil MN\rfloor} + \delta(\widetilde{\Xi}_3) B_{\mu\nu}{}^{\lceil MN\rfloor} + \delta(\widetilde{\Phi}_3) B_{\mu\nu}{}^{\lceil MN\rfloor} \\
&\qquad +2\Lambda_1^{[Q}\Lambda_2^{P]} \Big[ 8\eta_{\rho[\mu}\eta_{\nu]\sigma}\mathcal{I}_{\Lambda\Sigma}\, \mathbb{P}^{MN}{}_P{}^{\Lambda}\mathbb{P}^{RS}{}_Q{}^{\Sigma} \\
&\qquad -4\varepsilon_{\mu\nu\rho\sigma}\mathcal{R}_{\Lambda\Sigma}\, \mathbb{P}^{MN}{}_P{}^{\Lambda}\mathbb{P}^{RS}{}_Q{}^{\Sigma} \\
&\qquad -\varepsilon_{\mu\nu\rho\sigma}\left(\mathbb{P}^{MN}{}_P{}^{\Lambda}\mathbb{P}^{RS}{}_{Q\Lambda} + \mathbb{P}^{MN}{}_{P\Lambda}\mathbb{P}^{RS}{}_Q{}^{\Lambda}\right)\Big] \frac{\partial S_0}{\partial B_{\rho\sigma}{}^{\lceil RS\rfloor}} \,. \qquad (7.66)
\end{aligned}
$$

The parameters $\widetilde{\Lambda}_3$, $\widetilde{\Xi}_3$ and $\widetilde{\Phi}_3$ take the following values:

$$
\widetilde{\Lambda}_3^M = \Lambda_3^M \,, \; \widetilde{\Xi}_{3\mu}{}^{\lceil MN\rfloor} = \Xi_{3\mu}{}^{\lceil MN\rfloor} \,, \; \widetilde{\Phi}_{3\mu\nu}{}^{\lceil M\lceil NP\rfloor\rfloor} = \Lambda_1^{\lceil M}\mathcal{G}_{\mu\nu}{}^{\lceil N}\Lambda_2^{P\rfloor\rfloor} - (1 \leftrightarrow 2) \,. \qquad (7.67)
$$

This result is slightly more complicated and we note the following differences with (7.48):

- ▸ The parameter $\widetilde{\Phi}_3$ differs from $\Phi_3$, i.e. the field strengths $\mathcal{H}_{\mu\nu}{}^M$ have been replaced by their scalar dependent counterparts $\mathcal{G}_{\mu\nu}{}^M$.
- ▸ This in turn leads to a difference in the structure functions:

$$
\widetilde{\boldsymbol{T}}_{\mu\nu}{}^{M\lceil NP\rfloor}{}_{RS} = \delta_R{}^{\lceil M}\mathcal{G}_{\mu\nu}{}^{\lceil N}\delta_S{}^{P\rfloor\rfloor} - \delta_S{}^{\lceil M}\mathcal{G}_{\mu\nu}{}^{\lceil N}\delta_R{}^{P\rfloor\rfloor} \,. \qquad (7.68)
$$

  Again $\mathcal{H}_{\mu\nu}{}^M$ has been replace by $\mathcal{G}_{\mu\nu}{}^M$.
- ▸ Finally, the last three lines in equation (7.66) are proportional to the equations of motion. These terms fit into the general expression (7.56) with $\widetilde{\boldsymbol{E}}_{\mu\nu}{}^{MN}{}_{\rho\sigma}{}^{RS}{}_{PQ}$ different from zero:

$$
\begin{aligned}
&\widetilde{\boldsymbol{E}}_{\mu\nu}{}^{MN}{}_{\rho\sigma}{}^{RS}{}_{PQ} = \\
&\quad -16\eta_{\rho[\mu}\eta_{\nu]\sigma}\mathcal{I}_{\Lambda\Sigma}\, \mathbb{P}^{MN}{}_{[P}{}^{\Lambda}\mathbb{P}^{RS}{}_{Q]}{}^{\Sigma} + 8\varepsilon_{\mu\nu\rho\sigma}\mathcal{R}_{\Lambda\Sigma}\, \mathbb{P}^{MN}{}_{[P}{}^{\Lambda}\mathbb{P}^{RS}{}_{Q]}{}^{\Sigma} \\
&\quad +2\varepsilon_{\mu\nu\rho\sigma}\left(\mathbb{P}^{MN}{}_{[P}{}^{\Lambda}\mathbb{P}^{RS}{}_{Q]\Lambda} + \mathbb{P}^{MN}{}_{[P\Lambda}\mathbb{P}^{RS}{}_{Q]}{}^{\Lambda}\right) \,. \qquad (7.69)
\end{aligned}
$$

To summarize, let us repeat the main points of our discission. If the embedding tensor formalism is modified by the introduction of a Lagrangian, one is necessarily dealing with an open algebra. The general form of such an algebra is given in

(7.56), and we checked that this relation is indeed satisfied. At the same time, the calculations provided us with an expression for the $\widetilde{\boldsymbol{E}}$-tensors and structure constants. This lets us also conclude that we are dealing with a soft algebra, i.e. the structure constants depend on the fields in the theory.

Of course this is not the end of the story. Several higher order commutators need to be evaluated in order to define the full structure of the algebra. For example, at the second order we find the Jacobi identity,

$$[\delta_1,[\delta_2,\delta_3]]\phi^i + \text{cyclic in } 123 = 0\,, \tag{7.70}$$

which leads to extra relations between $\widetilde{\boldsymbol{R}}$, $\widetilde{\boldsymbol{T}}$ and $\widetilde{\boldsymbol{E}}$ due to (7.56). Moreover, it requires the introduction of several new tensors. In general, this process needs to be continued up to arbitrary order in the commutators, until it terminates. In this text, however, we will not go beyond first order since the most interesting properties of the algebra follow already from a single commutator on the fields.

### Zero modes

With the knowledge of the gauge generators from §7.2 and §7.3, we can now address the (in)dependence of the gauge transformations. Again we will distinguish between two cases:

1. For the formalism without an action, the question whether the gauge transformations $\{\delta(\Lambda),\, \delta(\Xi),\, \delta(\Phi)\}$ are (in)dependent, can be formulated as follows: do there exist vectors $\boldsymbol{Z}_{(\boldsymbol{1})}{}^{a_0}{}_{a_1}$, such that for all $i$

$$\boldsymbol{R}^i{}_{a_0}\boldsymbol{Z}_{(\boldsymbol{1})}{}^{a_0}{}_{a_1} = 0\,? \tag{7.71}$$

   The index $a_1$ enumerates the possible outcomes. If (7.71) has $m_1 \neq 0$ non-trivial solutions, then $a_1$ takes $m_1$ different values and it means that there exist $m_1$ dependencies between the gauge generators. In this case, the algebra is called reducible and the $\boldsymbol{Z}_{(\boldsymbol{1})}{}^{a_0}{}_{a_1}$ are its zero modes. If (7.71) has no non-trivial solutions, then the gauge transformations are independent and the algebra is called irreducible.

2. If the formalism is embedded into the framework with a classical action $S_0$, then we should consider the generators $\widetilde{\boldsymbol{R}}^i{}_{a_0}$ instead and equation (7.71) has to be modified to

$$\widetilde{\boldsymbol{R}}^i{}_{a_0}\widetilde{\boldsymbol{Z}}_{(\boldsymbol{1})}{}^{a_0}{}_{a_1} = \partial_j S_0\, \widetilde{\boldsymbol{V}}_{(\boldsymbol{1})}{}^{ji}{}_{a_1}\,, \tag{7.72}$$

   for some tensors $\widetilde{\boldsymbol{Z}}_{(\boldsymbol{1})}{}^{a_0}{}_{a_1}$ and $\widetilde{\boldsymbol{V}}_{(\boldsymbol{1})}{}^{ij}{}_{a_1} = -\widetilde{\boldsymbol{V}}_{(\boldsymbol{1})}{}^{ji}{}_{a_1}$. The right hand side of (7.72) is now proportional to the field equations, which means that the $\widetilde{\boldsymbol{Z}}_{(\boldsymbol{1})}{}^{a_0}{}_{a_1}$ are *on-shell* null vectors (or zero modes):

$$\widetilde{\boldsymbol{R}}^i{}_{a_0}\widetilde{\boldsymbol{Z}}_{(\boldsymbol{1})}{}^{a_0}{}_{a_1}\Big|_{\text{on-shell}} = 0\,. \tag{7.73}$$

The presence of $\widetilde{\boldsymbol{V}}_{\!(\mathbf{1})}{}^{ij}{}_{a_1}$ in (7.72) is a way to extend this statement off-shell. If $\{a_1\}$ is non-empty, then the gauge generators have $m_1$ on-shell dependencies and the algebra is reducible.

The strategy to solve (7.71) and (7.72) is to evaluate the different possibilities for the index $i$ and work out the summation over the $a_0$-index. Then one can try to find particular solutions (which is based on trial and error) that fix the precise form of the $a_1$-type indices. The details of this calculation will be omitted since they are not very enlightening; we refer to section 3 of our paper [3] where all the intermediate steps can be found. Here we only present the final result, which proves that the gauge generators in the $D = 4$ embedding tensor formalism with action $S_{\mathrm{VT}}$ are not all independent. For the indices $i$ and $a_0$ restricted to $\{\mu N,\, \mu\nu[\![NP]\!]\}$ and $\{Q,\, \mu[\![RS]\!],\, \mu\nu[\![Q[\![RS]\!]]\!]\}$ respectively, we checked that equation (7.72) have non-trivial solutions for the zero modes $\widetilde{\boldsymbol{Z}}_{(\mathbf{1})}{}^{a_0}{}_{a_1}$ and corresponding tensors $\widetilde{\boldsymbol{V}}_{\!(\mathbf{1})}{}^{ij}{}_{a_1}$. We found three solutions in total, for $a_1 = [\![KL]\!]$, $a_1 = \rho[\![K[\![LM]\!]]\!]$ and $a_1 = \rho\sigma[\![K[\![L[\![MN]\!]]\!]]\!]$. Then $\widetilde{\boldsymbol{Z}}_{(\mathbf{1})}{}^{a_0}{}_{a_1}$ is a $(3\times 3)$ block matrix; the rows are enumerated by $a_0$ and the columns by $a_1$:

$$\widetilde{\boldsymbol{Z}}_{(\mathbf{1})}{}^{a_0}{}_{a_1} = \begin{pmatrix} Y^Q{}_{KL} & 0 & 0 \\ D_\mu{}^{RS}{}_{KL} & Y^{RS}{}_{K\lceil LM\rfloor}\delta^\rho_\mu & 0 \\ -\mathcal{G}_{\mu\nu}{}^{\lceil Q}\mathbb{P}^{RS\rfloor}{}_{KL} & 2D_{[\mu}{}^{Q\lceil RS\rfloor}{}_{K\lceil LM\rfloor}\delta^\rho_{\nu]} & Y^{Q\lceil RS\rfloor}{}_{K\lceil L\lceil MN\rfloor\rfloor}\delta^{\rho\sigma}_{[\mu\nu]} \end{pmatrix}. \tag{7.74}$$

Only for the solution with $a_1 = [\![KL]\!]$ there is a non-vanishing $\widetilde{\boldsymbol{V}}_{\!(\mathbf{1})}$-tensor, given by

$$\widetilde{\boldsymbol{V}}_{\!(\mathbf{1})}{}^{ij}{}_{\lceil KL\rfloor} = \begin{pmatrix} 0 & 0 \\ 0 & -4\varepsilon_{\mu\nu\rho\sigma}\delta_{\lceil K}{}^{\lceil N}\Omega^{P\rfloor\lceil R}\delta_{L\rfloor}{}^{S\rfloor} \end{pmatrix}. \tag{7.75}$$

Some remarks about this solution are in order. First, we note that the lower right entry of the zero-mode matrix contains a tensor $Y^{Q\lceil RS\rfloor}{}_{K\lceil L\lceil MN\rfloor\rfloor}$, which is a generalization of $Y^Q{}_{KL}$ and $Y^{RS}{}_{K\lceil LM\rfloor}$. Its precise definition in terms of the other known tensors in the embedding tensor formalism is not important, but we do note the following important property:

$$Y^{TU}{}_{Q\lceil RS\rfloor}Y^{Q\lceil RS\rfloor}{}_{K\lceil L\lceil MN\rfloor\rfloor} = 0\,. \tag{7.76}$$

Moreover, we recognize a certain systematics in the solution for the zero modes: the diagonal entries of $\widetilde{\boldsymbol{Z}}_{(\mathbf{1})}{}^{a_0}{}_{a_1}$ are all proportional to a $Y$-tensor and the 21- and 32-elements contain a derivative. This special structure will be further investigated in the next section, where we show that $\widetilde{\boldsymbol{Z}}_{(\mathbf{1})}{}^{a_0}{}_{a_1}$ has non-maximal rank, which means that not all zero modes are independent. Finally, one can show that the solutions $\boldsymbol{Z}_{(\mathbf{1})}{}^{a_0}{}_{a_1}$ of (7.71) are identical to $\widetilde{\boldsymbol{Z}}_{(\mathbf{1})}{}^{a_0}{}_{a_1}$, except for the lower left entry, where $\mathcal{G}_{\mu\nu}{}^Q$ should be replaced by $\mathcal{H}_{\mu\nu}{}^Q$. Moreover, there are no $\boldsymbol{V}_{(\mathbf{1})}{}^{ij}{}_{a_1}$-tensors.

### Higher stage zero modes

If the 3 solutions for $\boldsymbol{Z}_{\mathbf{(1)}}{}^{a_0}{}_{a_1}$ or $\widetilde{\boldsymbol{Z}}_{\mathbf{(1)}}{}^{a_0}{}_{a_1}$ are independent, then the theory is called first-stage reducible. However, this may not happen; there can be 'level-two' gauge invariances that reflect the dependencies among the $\boldsymbol{Z}_{\mathbf{(1)}}{}^{a_0}{}_{a_1}$ or $\widetilde{\boldsymbol{Z}}_{\mathbf{(1)}}{}^{a_0}{}_{a_1}$. This reasoning can be repeated for the level-two generators, and it possibly leads to dependencies at higher stages. This brings us to the concept of an $L$-th stage reducible theory, which means that only at level $L$, all the generators are independent. In order to determine the level $L$ for the gauge structure of the embedding tensor formalism, we will investigate the dependencies among the zero modes in (7.74). We need to solve an equation that is similar to (7.71) or (7.72):

1. For the transformations in the absence of an action, we look for non-trivial tensors $\boldsymbol{Z}_{\mathbf{(2)}}{}^{a_1}{}_{a_2}$ that are solutions of

$$\boldsymbol{Z}_{\mathbf{(1)}}{}^{a_0}{}_{a_1}\boldsymbol{Z}_{\mathbf{(2)}}{}^{a_1}{}_{a_2} = 0\,. \tag{7.77}$$

   The index $a_2$ labels the $m_2$ different solutions and therefore the possible dependencies of the zero modes. The new tensors $\boldsymbol{Z}_{\mathbf{(2)}}{}^{a_1}{}_{a_2}$ are called 'zero modes for zero modes' or second stage zero modes.

2. In the presence of an action, equation (7.77) needs to be modified to

$$\widetilde{\boldsymbol{Z}}_{\mathbf{(1)}}{}^{a_0}{}_{a_1}\widetilde{\boldsymbol{Z}}_{\mathbf{(2)}}{}^{a_1}{}_{a_2} = \partial_i S_0\ \widetilde{\boldsymbol{V}}_{\mathbf{(2)}}{}^{i\,a_0}{}_{a_2}\,. \tag{7.78}$$

   The $\widetilde{\boldsymbol{Z}}_{\mathbf{(2)}}{}^{a_1}{}_{a_2}$ are $m_2$ on-shell null vectors of the zero modes. The tensors $\widetilde{\boldsymbol{V}}_{\mathbf{(2)}}{}^{i\,a_0}{}_{a_2}$ in (7.78) provide an off-shell extension of this statement.

We will look for non-trivial solutions of (7.78) with $\widetilde{\boldsymbol{Z}}_{\mathbf{(1)}}{}^{a_0}{}_{a_1}$ given in (7.74); the solutions of (7.77) will again be very similar. This time our strategy is to make a motivated guess for the form of the solutions, and then check whether (7.78) is indeed satisfied. From the previous section, we know that

$$\begin{aligned} a_0 &\in \{K_1, \quad \mu\llbracket K_1K_2\rrbracket, \quad \mu\nu\llbracket K_1\llbracket K_2K_3\rrbracket\rrbracket \quad \}\,, \\ a_1 &\in \{\llbracket K_1K_2\rrbracket,\ \mu\llbracket K_1\llbracket K_2K_3\rrbracket\rrbracket,\ \mu\nu\llbracket K_1\llbracket K_2\llbracket K_3K_4\rrbracket\rrbracket\rrbracket\}\,. \end{aligned} \tag{7.79}$$

Comparing these two index sets, we expect that this structure can be continued and

$$a_2 \in \{\llbracket K_1\llbracket K_2K_3\rrbracket\rrbracket,\ \mu\llbracket K_1\ldots K_4\rrbracket..\rrbracket,\ \mu\nu\llbracket K_1\ldots K_5\rrbracket..\rrbracket\}\,. \tag{7.80}$$

Therefore, we propose the following form for $\widetilde{\boldsymbol{Z}}_{\boldsymbol{(2)}}{}^{a_1}{}_{a_2}$, which looks very similar to the expression for the zero modes in (7.74):

$$\widetilde{\boldsymbol{Z}}_{\boldsymbol{(2)}}{}^{a_1}{}_{a_2} = ([A]\,[B]\,[C]) \quad \text{with} \quad [A] = \begin{bmatrix} -Y^{M_1M_2}{}_{K_1\lceil K_2K_3\rfloor} \\ D_\mu{}^{M_1\lceil M_2M_3\rfloor}{}_{K_1\lceil K_2K_3\rfloor} \\ \mathcal{G}_{\mu\nu}{}^{\lceil M_1}\mathbb{P}^{\ldots M_4\rfloor\rfloor}{}_{K_1\lceil K_2K_3\rfloor} \end{bmatrix},$$

$$[B] = \begin{bmatrix} 0 \\ -Y^{M_1\lceil M_2M_3\rfloor}{}_{K_1\lceil\ldots K_4\rfloor\rfloor}\delta^\rho_\mu \\ 2D_{[\mu}{}^{M_1\lceil\ldots M_4\rfloor\rfloor}{}_{K_1\lceil\ldots K_4\rfloor\rfloor}\delta^\rho_{\nu]} \end{bmatrix},\ [C] = \begin{bmatrix} 0 \\ 0 \\ -Y^{M_1\lceil\ldots M_4\rfloor\rfloor}{}_{K_1\lceil\ldots K_5\rfloor\ldots\rfloor}\delta^{[\rho\sigma]}_{[\mu\nu]} \end{bmatrix}. \tag{7.81}$$

On the diagonal, $Y$-tensors appear with an extra minus sign compared to the expression for the zero modes (7.74). The 21- and 32-elements contain a derivative and the lower left entry is proportional to the scalar dependent field strength.

In [3] we have computed the different entries in the matrix product $\widetilde{\boldsymbol{Z}}_{\boldsymbol{(1)}}{}^{a_0}{}_{a_1}\widetilde{\boldsymbol{Z}}_{\boldsymbol{(2)}}{}^{a_1}{}_{a_2}$ and showed that they are proportional to the field equations, just as we would expect from (7.78). This calculation was a check on the validity of (7.81) and at the same time, it provided us with an expression for $\widetilde{\boldsymbol{V}}_{\boldsymbol{(2)}}{}^{i\,a_0}{}_{a_2}$. We found that the only non-vanishing $\widetilde{\boldsymbol{V}}_{\boldsymbol{(2)}}$-tensor corresponds to the index combination $i = \rho\sigma\llbracket N_1N_2\rrbracket$, $a_0 = \mu\nu\llbracket M_1\llbracket M_2M_3\rrbracket\rrbracket$ and $a_2 = \llbracket K_1\llbracket K_2K_3\rrbracket\rrbracket$:

$$\begin{aligned}
&\widetilde{\boldsymbol{V}}_{\boldsymbol{(2)}\,\rho\sigma}{}^{\lceil N_1N_2\rfloor}{}_{\mu\nu}{}^{\lceil M_1\lceil M_2M_3\rfloor\rfloor}{}_{\lceil K_1\lceil K_2K_3\rfloor\rfloor} = \\
&\quad -2\varepsilon_{\mu\nu\rho\sigma}\Big(\delta_{\lceil K_1}{}^{\lceil N_1}\Omega^{N_2\rfloor P_1}\delta_{\lceil K_2}{}^{P_2}\delta_{K_3\rfloor\rfloor}{}^{P_3} \\
&\qquad\qquad +\delta_{\lceil K_1}{}^{P_1}\delta_{\lceil K_2}{}^{\lceil N_1}\Omega^{N_2\rfloor P_2}\delta_{K_3\rfloor\rfloor}{}^{P_3} \\
&\qquad\qquad +\delta_{\lceil K_1}{}^{P_1}\delta_{\lceil K_2}{}^{P_2}\delta_{K_3\rfloor\rfloor}{}^{\lceil N_1}\Omega^{N_2\rfloor P_3}\Big)\mathbb{P}^{M_1\lceil M_2M_3\rfloor}{}_{P_1\lceil P_2P_3\rfloor}\,.
\end{aligned} \tag{7.82}$$

In the end, we have proven that our proposal for $\widetilde{\boldsymbol{Z}}_{\boldsymbol{(2)}}{}^{a_1}{}_{a_2}$ in (7.81) is indeed a solution of (7.78) for each value of $a_2$. It means that the zero modes $\widetilde{\boldsymbol{Z}}_{\boldsymbol{(1)}}{}^{a_0}{}_{a_1}$ are not all independent and the gauge algebra is at least reducible up to level 2. The same conclusion holds for the zero modes in the embedding tensor formalism without an action, i.e. $\boldsymbol{Z}_{\boldsymbol{(1)}}{}^{a_0}{}_{a_1}$. These are also not independent and the solutions of (7.77) are identical to (7.81), except for the lower left entry, where the field strengths $\mathcal{G}_{\mu\nu}{}^M$ should be replaced by $\mathcal{H}_{\mu\nu}{}^M$.

Given these non-trivial expressions for the 1st and 2nd stage zero modes, one could wonder whether there exists a level for which this construction terminates. In other words, is there a level $s$, for which

$$\widetilde{\boldsymbol{Z}}_{\boldsymbol{(s-1)}}{}^{a_{s-2}}{}_{a_{s-1}}\widetilde{\boldsymbol{Z}}_{\boldsymbol{(s)}}{}^{a_{s-1}}{}_{a_s} = \partial_i S_0\,\widetilde{\boldsymbol{V}}_{\boldsymbol{(s)}}{}^{i\,a_{s-2}}{}_{a_s} \tag{7.83}$$

has *no* non-trivial solutions? If this is the case for a value $s = L$, then the theory is said to be $L$-th stage reducible.

So the question is whether there exists a finite value for $L$ in the embedding tensor formalism. A priori, there does not seem to be such a finite level at which the above construction comes to an end. Indeed, one can propose the following expressions for the zero modes and non-vanishing $\widetilde{\boldsymbol{V}}$-tensors at arbitrary level $s \geq 1$:

$$\widetilde{\boldsymbol{Z}}_{(\boldsymbol{s})}{}^{a_{s-1}}{}_{a_s} = \left([A_{(s)}]\,[B_{(s)}]\,[C_{(s)}]\right)\,, \tag{7.84}$$

with

$$[A_{(s)}] = \left[\begin{array}{c} (-1)^{s+1}\, Y^{M_1\lceil M_2\ldots M_s\rfloor ..\rfloor}{}_{N_1\lceil N_2\ldots N_{s+1}\rfloor ..\rfloor} \\ D_\mu{}^{M_1\lceil M_2\ldots M_{s+1}\rfloor ..\rfloor}{}_{N_1\lceil N_2\ldots N_{s+1}\rfloor ..\rfloor} \\ (-1)^s\, \mathcal{G}_{\mu\nu}{}^{\lceil M_1}\mathbb{P}^{M_2\lceil\ldots M_{s+2}\rfloor ..\rfloor}{}_{N_1\lceil N_2\ldots N_{s+1}\rfloor ..\rfloor} \end{array}\right]\,, \tag{7.85}$$

$$[B_{(s)}] = \left[\begin{array}{c} 0 \\ (-1)^{s+1}\, Y^{M_1\lceil M_2\ldots M_{s+1}\rfloor ..\rfloor}{}_{N_1\lceil N_2\ldots N_{s+2}\rfloor ..\rfloor}\delta^\rho_\mu \\ 2D_{[\mu}{}^{M_1\lceil M_2\ldots M_{s+2}\rfloor ..\rfloor}{}_{N_1\lceil N_2\ldots N_{s+2}\rfloor ..\rfloor}\delta^\rho_{\nu]} \end{array}\right]\,, \tag{7.86}$$

$$[C_{(s)}] = \left[\begin{array}{c} 0 \\ 0 \\ (-1)^{s+1}\, Y^{M_1\lceil M_2\ldots M_{s+2}\rfloor ..\rfloor}{}_{N_1\lceil N_2\ldots N_{s+3}\rfloor ..\rfloor}\delta^{[\rho\sigma]}_{[\mu\nu]} \end{array}\right]\,, \tag{7.87}$$

and

$$\begin{aligned} &\widetilde{\boldsymbol{V}}_{(\boldsymbol{s})\,\rho\sigma}{}^{\lceil N_1 N_2\rfloor}{}_{\mu\nu}{}^{M_0\lceil M_1\ldots M_s\rfloor ..\rfloor}{}_{K_0\lceil K_1\ldots K_s\rfloor ..\rfloor} = \\ &-2\varepsilon_{\mu\nu\rho\sigma}\Big(\delta_{\lceil K_0}{}^{\lceil N_1}\Omega^{N_2\rfloor P_0}\delta_{\lceil K_1}{}^{P_1}\ldots\delta_{K_s\rfloor ..\rfloor}{}^{P_s} \\ &+\delta_{\lceil K_0}{}^{P_0}\delta_{\lceil K_1}{}^{\lceil N_1}\Omega^{N_2\rfloor P_1}\delta_{\lceil K_2}{}^{P_2}\ldots\delta_{K_s\rfloor ..\rfloor}{}^{P_s} \\ &+\ldots \\ &+\delta_{\lceil K_0}{}^{P_0}\ldots\delta_{K_s\rfloor ..\rfloor}{}^{\lceil N_1}\Omega^{N_2\rfloor P_s}\Big)\mathbb{P}^{M_0\lceil M_1\ldots M_s\rfloor ..\rfloor}{}_{P_0\lceil P_1\ldots P_s\rfloor ..\rfloor}\,. \end{aligned} \tag{7.88}$$

Equation (7.83) is always satisfied for this combination of tensors, irrespective of the value of $s$. Therefore, we conclude that there always exists a zero mode at every arbitrary level and the theory is infinitely reducible.

Of course, this is just a formal statement since in particular examples, one needs to evaluate the different projection operators for the special brackets in (7.84)-(7.88). For certain choices of the embedding tensor, the projectors $\mathbb{P}^{M_1\lceil M_2\ldots M_p\rfloor ..\rfloor}{}_{N_1\lceil N_2\ldots N_p\rfloor ..\rfloor}$ might vanish for $p$ bigger than a certain value, say $\ell$. This means that also the corresponding objects in (7.84)-(7.88) with more than $\ell$ upper or lower indices are identically zero. Therefore, $\ell$ determines the level, $L$, at which the zero modes of the algebra become independent. We conclude that a case-by-case study is needed to determine $L$ and as such, no general statement can be made about its value.

We have now come to the end of our discussion on the gauge structure of the embedding tensor formalism with 1- and 2-forms and local transformations $\delta(\Lambda)$, $\delta(\Xi)$ and $\delta(\Phi)$. We found an algebra that is open in general, with field dependent structure functions and a hierarchy of zero modes that has no obvious ending. The details of this gauge structure are contained in a large set of tensors, such as the gauge generators, structure functions, zero modes, etc. These are complicated expressions of the fields and the embedding tensor, which makes it hard to take them into account in explicit calculations. Therefore, one might wonder whether there exists an underlying prescription that provides a unified picture for these complicated tensors. In the next section we will see that such a unifying formalism does exist and that all the gauge structure tensors naturally fit into one 'master equation'.

## 7.4 The field-antifield formalism

The formalism that we have in mind is the field-antifield or Batalin-Vilkovisky (BV) formalism. From the introduction we recall that this formalism was originally introduced as an extension of the Faddeev-Popov procedure to quantize a broader class of field theories with local symmetries. It is particularly useful for theories with a complicated gauge structure such as open, soft and/or reducible algebras. In the previous sections we saw that the embedding tensor formalism falls into this class and the BV formalism therefore provides all the tools for its quantization. However, we will not pursue this quantization, but rather concentrate on how the (classical) embedding tensor formalism fits into the structure of the classical BV formalism.

To this end, we introduce in §7.4 all the ingredients that make up the classical BV formalism. Then, in § 7.4, we will see how the embedding tensor formalism fits into this framework and how the BV formalism provides a simplified description for its complicated tensor structure.

## Classical BV theory

Consider a classical system described by the action[8] $S_0[\phi]$ that is a functional of the bosonic fields $\phi^i$. This means that the fields have even parity, i.e.

$$\epsilon[\phi^i] = 0\,. \tag{7.89}$$

In general the classical action $S_0$ can also contain fermionic degrees of freedom, but this case will not be considered here.

The theory has $m_0$ bosonic gauge symmetries that are generated by $\widetilde{\boldsymbol{R}}^i{}_{a_0}$ and have corresponding local parameters $\varepsilon^{a_0}$. This then leads to $m_0$ Noether identities as in (7.53), an expression for the gauge commutators as in (7.56), $m_1$ zero modes as solutions of (7.72), etc. All these equations are written down in terms of certain tensors that determine the complete gauge structure of the theory. The main purpose of the classical BV-formalism is to provide a consistent framework that incorporates all these tensors in a transparent way. In particular, this is achieved through the construction of a new action, denoted by $S$, which is an extension of the classical action $S_0$. In brief, the construction of $S$ involves five steps, each of which will be discussed in more detail later on.

1. Ghost fields are introduced to compensate for the gauge degrees of freedom. When dealing with a reducible system (in which the gauge transformations are not all independent), also higher stage ghost fields need to be introduced. The original configuration space, consisting of the $\phi^i$, is enlarged to include these ghost fields, ghosts for ghosts, etc..

2. For each field, thus also for the (higher stage) ghost fields, an antifield is introduced.

3. On the space of fields and antifields, one defines an odd symplectic structure $(\,.\,,\,.\,)$, called the antibracket.

4. The classical action $S_0$ is extended to include terms involving fields and antifields and is denoted by $S$. It has to satisfy certain boundary conditions, such as the requirement that in the limit where all antifields are put to zero, the extended action $S$ reduces to $S_0$.

5. Finally, one imposes the classical master equation, $(S, S) = 0$. One finds solutions $S$ to this equation, subject to the appropriate boundary conditions. It turns out that these solutions are an expansion in the antifields and that the coefficients in the expansion are exactly the tensors that determine the gauge structure of the theory.

[8] In the previous sections about the embedding tensor formalism, we distinguished between the gauge algebra in the presence and in the absence of an action. To introduce the BV formalism we start with a system determined by an action $S_0$. Once we have introduced the formalism it is easy to consider the case where there is no gauge-invariant classical action.

Let us now consider each of these steps in more detail.

**Ghosts.** Suppose we are dealing with an irreducible theory with $m_0$ gauge invariances and corresponding parameters $\varepsilon^{a_0}$. Then at the quantum level $m_0$ ghost fields ${c_{(0)}}^{a_0}$ are needed, i.e. one for each parameter. However, for our purposes it is useful to introduce these ghost fields already at the classical level. Hence, the complete set of classical fields is $\chi^n = \{\phi^i, {c_{(0)}}^{a_0}\}$.

In a reducible theory the $m_0$ gauge invariances are not all independent; there exist zero modes for the gauge invariances. In principle these zero modes imply that we have introduced too many gauge parameters, but that can be necessary in order to preserve the covariance or locality of the theory. If there are $m_1$ first-level zero modes then one adds the ghost-for-ghost fields ${c_{(1)}}^{a_1}$ $(a_1 = 1, \cdots, m_1)$ to the above set $\chi^n$. In general for an $L$-stage reducible theory, the set of fields $\chi^n$ (where $n = 1, \cdots, N$) is

$$\chi^n = \left\{\phi^i, {c_{(s)}}^{a_s};\, s = 0, \cdots, L\,;\, a_s = 1, \cdots, m_s\right\}. \tag{7.90}$$

The ghosts are defined as having opposite statistics to the corresponding gauge parameter, ghost for ghosts as having the same statistics as the gauge parameter, and so on, with the statistics alternating for higher level ghosts. We can write this as

$$\epsilon[{c_{(s)}}^{a_s}] = (s+1) \text{mod } 2, \tag{7.91}$$

where $\epsilon[{c_{(s)}}^{a_s}]$ denotes the parity of the (higher stage) ghost. Moreover, an additive conserved charge, called ghost number $\text{gh}[\chi^n]$, is assigned to each of these fields $\chi^n$. The classical fields $\phi^i$ have ghost number zero, whereas ordinary ghosts have ghost number one. Ghost for ghosts (first level ghosts), have ghost number two etc.

$$\text{gh}[\phi^i] = 0\,, \qquad \text{gh}[{c_{(s)}}^{a_s}] = s+1\,. \tag{7.92}$$

**Antifields.** Next, one introduces an antifield $\chi^*_n$ $(n = 1, \cdots, N)$ for each field $\chi^n$. These antifields should be thought of as a mathematical tool to set up the formalism. The ghost number and statistics of $\chi^*_n$ are

$$\text{gh}[\chi^*_n] \;=\; -\,\text{gh}[\chi^n] - 1\,, \tag{7.93}$$

$$\epsilon[\chi^*_n] \;=\; (\epsilon[\chi^n] + 1) \,\text{mod } 2\,, \tag{7.94}$$

such that $\chi^n$ and $\chi^*_n$ have opposite statistics. In the future, we will denote the total set of fields and antifields[9] with $z^a = \{\chi^n, \chi^*_n\}$. For each Field, we introduce

[9] In order to refer to the fields and antifields simultaneously, we will use the terminology 'Fields', with a capital letter F.

an antifield number $afn[z^a]$ which will become important later on.

$$afn[z^a] = \begin{cases} 0 : & \mathrm{gh}[z^a] \geq 0\,, \\ -\mathrm{gh}[z^a] : & \mathrm{gh}[z^a] < 0\,. \end{cases} \tag{7.95}$$

**The antibracket.** On the space of Fields one introduces an odd symplectic structure, the antibracket $(\,.\,,\,.\,)$. It is defined by

$$(X,Y) \equiv \frac{\partial_r X}{\partial \chi^n}\frac{\partial_l Y}{\partial \chi^*_n} - \frac{\partial_r X}{\partial \chi^*_n}\frac{\partial_l Y}{\partial \chi^n}, \tag{7.96}$$

where the subscripts $\partial_r$ and $\partial_l$ denote right and left differentiation respectively, and $X$ and $Y$ are arbitrary functionals of the Fields $z^a$. In the case where $X$ and $Y$ are bosonic quantities, the antibracket has the following useful properties. It is similar to the Poisson bracket, but symmetric under the exchange of $X$ and $Y$. It has odd statistics, i.e. $\epsilon[(X,Y)] = 1$. Moreover, the bracket of two identical bosonic functionals $X$ of the Fields is

$$(X,X) = 2\frac{\partial_r X}{\partial \chi^n}\frac{\partial_l X}{\partial \chi^*_n}. \tag{7.97}$$

Finally we note that the definition in (7.96) can also be written as

$$(X,Y) = \frac{\partial_r X}{\partial z^a}\Omega^{ab}\frac{\partial_l Y}{\partial z^b}\,, \qquad \text{where} \quad \Omega^{ab} \equiv \begin{pmatrix} 0 & \delta^n_m \\ -\delta^n_m & 0 \end{pmatrix}, \tag{7.98}$$

which is why we call the antibracket a symplectic structure.

**The extended action and boundary conditions.** Let $S[\chi,\chi^*]$ be an arbitrary functional of the Fields with the dimension of an action, even parity $\epsilon[S] = 0$, and zero ghost number $\mathrm{gh}[S] = 0$. This functional is called an *extended action* if it satisfies the following boundary conditions:

(i) In the 'classical limit', $S$ reduces to $S_0$,

$$S[\chi,\chi^*]|_{\chi^*_n=0} \;=\; S_0[\phi^i]\,, \tag{7.99}$$

i.e. when all the antifields are put to zero, the extended action reduces to the original action $S_0$. This requirement means that $S$ can be written as an expansion in the antifields, with the classical action $S_0$ at zeroth order:

$$S[\chi,\chi^*] = S_0 + \text{terms that are linear, quadratic,... in the antifields.} \tag{7.100}$$

This can be made more precise if we order all the terms in $S$ according to their antifield number:

$$S = \sum_k S_k = S_0 + S_1 + S_2 + \cdots \,, \tag{7.101}$$

where $afn(S_i) = i$. An expression for $S_i$ for the lowest orders of $i$ will be given in due course.

(ii) The second boundary condition that should be satisfied is more technical. It is called the properness condition and it takes the following form:

$$\text{rank} \frac{\partial_l \partial_r S}{\partial z^a \partial z^b}\bigg|_\Sigma = N, \tag{7.102}$$

where $N$ is the number of fields $\chi^n$ or antifields $\chi^*_n$ and $\Sigma$ denotes the subspace of stationary points in the space of Fields,

$$z_0^a \in \Sigma \quad \longleftrightarrow \quad \frac{\partial_r S}{\partial z^a}\bigg|_{z_0^a} = 0\,. \tag{7.103}$$

Condition (7.102) tells us that the rank of the matrix $\frac{\partial_l \partial_r S}{\partial z^a \partial z^b}$ is half its dimensions. Due to (7.105) in the next paragraph, this is the maximum that can be achieved and it guarantees that all the symmetries in the theory have been taken care of via the introduction of ghosts, zero modes and their antifields.

If we take into account the boundary conditions, the expansion in (7.101) looks like

$$\begin{aligned}
S_0 &= S_0[\phi]\,, \qquad\qquad (7.104)\\
S_1 &= \phi^*_i \widetilde{\boldsymbol{R}}^i{}_{a_0} {c_{(0)}}^{a_0}\,,\\
S_2 &= c^*_{(0)a_0} \left( \widetilde{\boldsymbol{Z}}_{\mathbf{(1)}}{}^{a_0}{}_{a_1} {c_{(1)}}^{a_1} + \frac{1}{2} \widetilde{\boldsymbol{T}}^{a_0}{}_{b_0 c_0} {c_{(0)}}^{c_0} {c_{(0)}}^{b_0} \right)\\
&\quad + \phi^*_i \phi^*_j \left( \frac{1}{2} \widetilde{\boldsymbol{V}}_{\mathbf{(1)}}{}^{ji}{}_{a_1} {c_{(1)}}^{a_1} + \frac{1}{4} \widetilde{\boldsymbol{E}}^{ij}{}_{a_0 b_0} {c_{(0)}}^{a_0} {c_{(0)}}^{b_0} \right)\,,\\
S_3 &= c^*_{(1)a_1} \left( \widetilde{\boldsymbol{Z}}_{\mathbf{(2)}}{}^{a_1}{}_{a_2} {c_{(2)}}^{a_2} + \ldots + \frac{1}{2} \widetilde{\boldsymbol{F}}^{a_1}{}_{e_0 d_0 b_0} {c_{(0)}}^{b_0} {c_{(0)}}^{d_0} {c_{(0)}}^{e_0} \right)\\
&\quad + c^*_{(0)a_0} \phi^*_i \left( \widetilde{\boldsymbol{V}}_{\mathbf{(2)}}{}^{ia_0}{}_{a_2} {c_{(2)}}^{a_2} + \ldots - \frac{1}{2} \widetilde{\boldsymbol{D}}^{ia_0}{}_{e_0 d_0 b_0} {c_{(0)}}^{b_0} {c_{(0)}}^{d_0} {c_{(0)}}^{e_0} \right) + \ldots
\end{aligned}$$

Here we have written down the terms up to antifield number 3. The objects $\widetilde{\boldsymbol{R}}$, $\widetilde{\boldsymbol{Z}}_{\boldsymbol{(1)}}$, $\widetilde{\boldsymbol{T}}$, $\widetilde{\boldsymbol{V}}_{\boldsymbol{(1)}}$, $\widetilde{\boldsymbol{E}}$, $\widetilde{\boldsymbol{Z}}_{\boldsymbol{(2)}}$, $\widetilde{\boldsymbol{F}}$, $\widetilde{\boldsymbol{V}}_{\boldsymbol{(2)}}$ and $\widetilde{\boldsymbol{D}}$ should be thought of as generic functionals of the fields $\phi^i$ (not of the ghosts and antifields!) with a particular index structure. The dots in $S_3$ denote more tensors with different index structures, which we do not write explicitly here to clarify the discussion. Also note that each term in (7.104) has ghost number zero and even parity, and that the classical limit (7.99) is satisfied.

In the next paragraph, we will impose an extra equation on the extended action $S$, which will lead to a particular form for the different tensors.

**The classical master equation and general solutions.** The equation that we will impose is called the classical 'master equation' and it takes the form

$$(S,S)=0\,, \tag{7.105}$$

where $S$ is the extended action that we introduced in (7.101) and that satisfies the boundary conditions (7.99) and (7.102). Using (7.97), the master equation can also be written as

$$2\frac{\partial_r S}{\partial\chi^n}\frac{\partial_l S}{\partial\chi^*_n}=0\,. \tag{7.106}$$

To see what this equation really means, we plug in the expansion (7.101) into the left hand side of (7.105). We get

$$\begin{aligned}
(S,S) &= 2\frac{\partial_r S}{\partial\chi^n}\frac{\partial_l S}{\partial\chi^*_n}\\
&= 2\frac{\partial S_0}{\partial\phi^j}\widetilde{\boldsymbol{R}}^j_{a_0}{c_{(0)}}^{a_0}\\
&\quad+\phi^*_i\left(2\frac{\partial\widetilde{\boldsymbol{R}}^i_{a_0}}{\partial\phi^j}\widetilde{\boldsymbol{R}}^j_{b_0}-\widetilde{\boldsymbol{R}}^i_{c_0}{\widetilde{\boldsymbol{T}}^{c_0}}_{a_0b_0}+\frac{\partial S_0}{\partial\phi^j}{\widetilde{\boldsymbol{E}}^{ji}}_{a_0b_0}\right){c_{(0)}}^{a_0}{c_{(0)}}^{b_0}\\
&\quad+2\phi^*_i\left({\widetilde{\boldsymbol{R}}^i}_{a_0}{{\widetilde{\boldsymbol{Z}}_{\boldsymbol{(1)}}}^{a_0}}_{a_1}-\frac{\partial S_0}{\partial\phi^j}{{\widetilde{\boldsymbol{V}}_{\boldsymbol{(1)}}}^{ji}}_{a_1}\right){c_{(1)}}^{a_1}+\dots\\
&\quad+2c^*_{(0)a_0}\left({{\widetilde{\boldsymbol{Z}}_{\boldsymbol{(1)}}}^{a_0}}_{a_1}{{\widetilde{\boldsymbol{Z}}_{\boldsymbol{(2)}}}^{a_1}}_{a_2}-\frac{\partial S_0}{\partial\phi^j}{{\widetilde{\boldsymbol{V}}_{\boldsymbol{(2)}}}^{ja_0}}_{a_2}\right){c_{(2)}}^{a_2}+\dots
\end{aligned} \tag{7.107}$$

for the first few terms. To demand that this expression is zero (which is the content of the master equation), means that all the different terms in (7.107) should vanish separately. We see that the vanishing of the first, second and third term is equivalent with equations (7.53), (7.56) and (7.72) respectively. So it is clear that the master equation is satisfied up to antifield number 1 when we identify $\widetilde{\boldsymbol{R}}$, $\widetilde{\boldsymbol{T}}$, $\widetilde{\boldsymbol{E}}$, $\widetilde{\boldsymbol{Z}}_{(\mathbf{1})}$ and $\widetilde{\boldsymbol{V}}_{(\mathbf{1})}$ in (7.104) with the ones in §7.3. In other words, the master equation demands that the $\widetilde{\boldsymbol{R}}$ are exactly the gauge generators, the $\widetilde{\boldsymbol{T}}$ are the structure constants, the $\widetilde{\boldsymbol{Z}}_{(\mathbf{1})}$ are the first stage zero modes etc.

This discussion can be continued to terms with higher antifield number. For example, the last line in (7.107) vanishes if also the $\widetilde{\boldsymbol{Z}}_{(\mathbf{2})}$ and $\widetilde{\boldsymbol{V}}_{(\mathbf{2})}$ tensors are identified with the ones in §7.3. Eventually, due to the uniqueness of the solution $S$ (see [132, 138–140] for a proof), we conclude that the dots in (7.107) lead to *all* the relations that determine the gauge structure. For example, at higher order we will also discover the higher order zero modes (7.83).

To summarize, we have seen that the unique solution $S$ of the master equation (7.105), supplemented by the boundary conditions (7.99) and (7.102), is an expansion in the antifields that contains all the gauge structure tensors of the theory as its expansion coefficients. It is in this sense that all the details of the gauge structure of the theory are contained in one equation and that the BV-formalism provides a concise framework for the complicated properties of the gauge algebra.

Before we apply this strong result to our example of the embedding tensor formalism, let us make a final remark about gauge theories without an action. In §7.3 we have encountered an example of a consistent gauge algebra that closes on the fields, but without the existence of a Lagrangian description for these fields. Since our discussion on the BV formalism explicitly assumes the existence of a classical action $S_0$, one might wonder whether the case without an action can also be incorporated. This turns out to be possible if one makes the following modifications to the original formulation of the BV formalism. First, we set $S_0 = 0$, so there is no zeroth order term in the extended action $S$. Due to the absence of $S_0$, the proof of the uniqueness of the solution for S breaks down.[10] As a consequence the terms with $\phi^*_i \phi^*_j$ in $S_2$ are undetermined (as well as several other terms at higher order in the antifields). We can delete these terms, and we find a solution without any terms quadratic in antifields. In turn this leads to the vanishing of all terms in (7.107) that are proportional to the field equations. In other words, $(S,S)$ reduces to

$$\begin{aligned}(S,S) \quad &= \quad \phi^*_i \left( 2\frac{\partial \boldsymbol{R}^i_{a_0}}{\partial \phi^j} \boldsymbol{R}^j_{b_0} - \boldsymbol{R}^i_{c_0} \boldsymbol{T}^{c_0}{}_{a_0 b_0} \right) c_{(0)}{}^{a_0} c_{(0)}{}^{b_0} \\ &\qquad + 2\phi^*_i \boldsymbol{R}^i{}_{a_0} \boldsymbol{Z}_{(\mathbf{1})}{}^{a_0}{}_{a_1} c_{(1)}{}^{a_1} + 2c^*_{(0)a_0} \boldsymbol{Z}_{(\mathbf{1})}{}^{a_0}{}_{a_1} \boldsymbol{Z}_{(\mathbf{2})}{}^{a_1}{}_{a_2} c_{(2)}{}^{a_2} + \ldots \,,\end{aligned} \tag{7.108}$$

where we have also removed all the tildes in order to distinguish these tensors from the ones in the presence of an action. Again, if we impose the master equation, we encounter the relations (7.36), (7.71), (7.77) etc. which determine all the properties of the gauge structure.

[10] The Koszul-Tate differential is no more acyclic.

| | | | | parity | ghost $\sharp$ |
|---|---|---|---|---|---|
| fields $\phi^i$ | | $A_\mu{}^M$ | $B_{\mu\nu}{}^{\lceil MN\rfloor}$ | $+$ | 0 |
| ghosts $c_{(0)}{}^{a_0}$ | $c_{(0)}{}^M$ | $c_{(0)\mu}{}^{\lceil MN\rfloor}$ | $c_{(0)\mu\nu}{}^{\lceil M\lceil NP\rfloor\rfloor}$ | $-$ | 1 |
| $c_{(1)}{}^{a_1}$ | $c_{(1)}{}^{\lceil MN\rfloor}$ | $c_{(1)\mu}{}^{\lceil M\lceil NP\rfloor\rfloor}$ | $c_{(1)\mu\nu}{}^{\lceil M\lceil N\lceil PQ\rfloor\rfloor\rfloor}$ | $+$ | 2 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

**Table 7.2:** Field content of the BV formalism.

### Embedding tensor formalism and the BV formulation

Let us now apply the results form the previous section to the embedding tensor formalism. The field content can easily be identified and is summarized in Table 7.2. Each of the fields in this table gets a corresponding antifield $\phi^*_i$, $c^*_{(0)a_0}$, etc. The ghost number, parity and antifield number of the antifields can be determined via (7.93), (7.94) and (7.95) respectively.

Then we construct the extended action and impose the master equation. From our considerations of the previous section, we know that $S$ is given by the expansion in (7.101). However, the precise form of the $S_i$ depends on whether we consider the embedding tensor formalism in the absence or the presence of an action, i.e., in terms of the untilded or tilded tensors respectively. In the first case, the leading terms in the extended action are

$$
\begin{aligned}
S \;=\;& \phi^*_i \boldsymbol{R}^i{}_{a_0} c_{(0)}{}^{a_0} + c^*_{(0)a_0} \boldsymbol{Z_{(1)}}{}^{a_0}{}_{a_1} c_{(1)}{}^{a_1} + c^*_{(1)a_1} \boldsymbol{Z_{(2)}}{}^{a_1}{}_{a_2} c_{(2)}{}^{a_2} + \ldots \\
=\;& A^{*\,\mu}{}_M \left( D_\mu{}^M{}_K\, c_{(0)}{}^K - Y^M{}_{KL}\, c_{(0)\,\mu}{}^{\lceil KL \rfloor} \right) \\
& + B^{*\mu\nu}{}_{\lceil MN \rfloor} \Big( -2\mathcal{H}_{\mu\nu}{}^{\lceil M}\, c_{(0)}{}^{N\rfloor} + 2 D_{[\mu}{}^{MN}{}_{KL}\, c_{(0)\,\nu]}{}^{\lceil KL \rfloor} \\
& \qquad\qquad - Y^{MN}{}_{K\lceil LR \rfloor}\, c_{(0)\,\mu\nu}{}^{\lceil K \lceil LR \rfloor\rfloor} \Big) \\
& + \ldots \\
& + c^*_{(0)M} \left( Y^M{}_{KL}\, c_{(1)}{}^{\lceil KL \rfloor} \right) \\
& + c^*_{(0)}{}^{\mu}{}_{\lceil MN \rfloor} \left( D_\mu{}^{MN}{}_{KL}\, c_{(1)}{}^{\lceil KL \rfloor} + Y^{MN}{}_{K\lceil LR \rfloor}\, c_{(1)\mu}{}^{\lceil K\lceil LR \rfloor\rfloor} \right) \\
& + c^*_{(0)}{}^{\mu\nu}{}_{\lceil M \lceil NP \rfloor\rfloor} \Big( -\mathcal{H}_{\mu\nu}{}^{\lceil M} \mathbb{P}^{NP\rfloor}{}_{KL}\, c_{(1)}{}^{\lceil KL \rfloor} \\
& \qquad\qquad + 2 D_{[\mu}{}^{M\lceil NP \rfloor}{}_{K\lceil LR \rfloor}\, c_{(1)\nu]}{}^{\lceil K \lceil LR \rfloor\rfloor} \\
& \qquad\qquad + Y^{M\lceil NP \rfloor}{}_{K\lceil L\lceil RS \rfloor\rfloor}\, c_{(1)\,\mu\nu}{}^{\lceil K\lceil L\lceil RS \rfloor\rfloor\rfloor} \Big) \\
& + \ldots \\
& + c^*_{(1)\lceil MN \rfloor} \left( -Y^{MN}{}_{K\lceil LR \rfloor}\, c_{(2)}{}^{\lceil K\lceil LR \rfloor\rfloor} \right) \\
& + c^*_{(1)}{}^{\mu}{}_{\lceil M\lceil NP \rfloor\rfloor} \Big( D_\mu{}^{\lceil M\lceil NP \rfloor\rfloor}{}_{K\lceil LR \rfloor}\, c_{(2)}{}^{\lceil K\lceil LR \rfloor\rfloor} \\
& \qquad\qquad - Y^{M\lceil NP \rfloor}{}_{K\lceil L\lceil RS \rfloor\rfloor}\, c_{(2)\,\mu\nu}{}^{\lceil K\lceil L\lceil RS \rfloor\rfloor\rfloor} \Big) \\
& + \ldots
\end{aligned}
\tag{7.109}
$$

We only wrote down the covariant terms and we recognize a certain systematics in this expression. At each level in the antifields, we encounter the same objects $Y$, $D_\mu$ and $\mathcal{H}$ between the brackets, but multiplied by different ghost fields. This is due to the particular form of the gauge transformations and (higher order) zero modes. Also for higher levels in the antifields, we expect that this structure survives. The dots in (7.109) denote extra terms that contain non-covariant objects and higher orders in the antifields.

The same calculation can be done for the embedding tensor formalism in the presence of an action. Then the extended action $S$ contains extra terms, starting with the classical action $S_0$ at zeroth order. Also other new terms are present at higher orders in the antifields, as can be seen from (7.104). Since we identified the expansion coefficients ($\widetilde{\boldsymbol{R}}$, $\widetilde{\boldsymbol{T}}$, $\widetilde{\boldsymbol{E}}$, etc.) in (7.104) with the gauge structure tensors in §7.3, the latter can be substituted into the expressions for $S_i$ in (7.104). We will not do this again, since the final result looks very similar to (7.109).

All in all, we have shown that the BV formalism provides a very appropriate description for the complicated gauge structure of the embedding tensor formalism. It suffices to consider the extended action $S$ and assume the master equation $(S,S)=0$, in order to have a full handle on the gauge structure of the theory. To finish this section, we will further illustrate this by means of an example. We will consider the terms in the master equation that are proportional to $c^*_{(0)\,a_0}c_{(0)}{}^{b_0}c_{(0)}{}^{d_0}c_{(0)}{}^{e_0}$, and show that they give rise to the modified Jacobi identity. We start from

$$\begin{aligned}
(S,S) &= \ldots + 2\frac{\partial_r S_2}{\partial\phi^i}\frac{\partial_l S_1}{\partial\phi^*_i} + 2\frac{\partial_r S_2}{\partial c_{(0)}{}^{a_0}}\frac{\partial_l S_2}{\partial c^*_{(0)\,a_0}} + 2\frac{\partial_r S_2}{\partial c_{(1)}{}^{a_1}}\frac{\partial_l S_3}{\partial c^*_{(1)\,a_1}} + \ldots \\
&= c^*_{(0)\,a_0}\left(\frac{\partial\widetilde{\boldsymbol{T}}^{a_0}{}_{b_0d_0}}{\partial\phi^i}\widetilde{\boldsymbol{R}}^i{}_{e_0} + \widetilde{\boldsymbol{T}}^{a_0}{}_{b_0c_0}\widetilde{\boldsymbol{T}}^{c_0}{}_{d_0e_0}\right. \\
&\qquad \left. + \widetilde{\boldsymbol{Z}}_{\mathbf{(1)}}{}^{a_0}{}_{a_1}\widetilde{\boldsymbol{F}}^{a_1}{}_{e_0d_0b_0}\right)c_{(0)}{}^{b_0}c_{(0)}{}^{d_0}c_{(0)}{}^{e_0} + \ldots
\end{aligned} \tag{7.110}$$

Note that in the absence of a classical action $S_0$ in the embedding tensor formalism, the expression (7.110) is completely analogous, except that tilded tensors should be replaced by untilded ones.

If the master equation is satisfied, the terms that are proportional to $c^*_{(0)\,a_0}c_{(0)}{}^{b_0}c_{(0)}{}^{d_0}c_{(0)}{}^{e_0}$ should vanish, i.e.

$$\begin{aligned}
& c^*_{(0)\,a_0}\left(\frac{\partial\widetilde{\boldsymbol{T}}^{a_0}{}_{b_0d_0}}{\partial\phi^i}\widetilde{\boldsymbol{R}}^i{}_{e_0} + \widetilde{\boldsymbol{T}}^{a_0}{}_{b_0c_0}\widetilde{\boldsymbol{T}}^{c_0}{}_{d_0e_0} + \widetilde{\boldsymbol{Z}}_{\mathbf{(1)}}{}^{a_0}{}_{a_1}\widetilde{\boldsymbol{F}}^{a_1}{}_{e_0d_0b_0}\right)c_{(0)}{}^{b_0}c_{(0)}{}^{d_0}c_{(0)}{}^{e_0} \\
&\quad = 0
\end{aligned} \tag{7.111}$$

This imposes several relations between the gauge generators $\widetilde{\boldsymbol{R}}$, structure functions $\widetilde{\boldsymbol{T}}$, zero modes $\widetilde{\boldsymbol{Z}}_{\mathbf{(1)}}$ and tensors $\widetilde{\boldsymbol{F}}$. Let us calculate the easiest contribution, i.e. for the indices $a_0$, $b_0$, $d_0$, $e_0 \in \{K, L, M, \ldots\}$, and plug in the expressions for the structure functions and zero modes,

$$\begin{aligned}
& c^*_{(0)\,K}\left(\frac{\partial X_{[LM]}{}^K}{\partial\phi^i}\widetilde{\boldsymbol{R}}^i{}_N + X_{[LP]}{}^K X_{[MN]}{}^P + Y^K{}_{PQ}\widetilde{\boldsymbol{F}}^{\lceil PQ\rfloor}{}_{NML}\right)c_{(0)}{}^L c_{(0)}{}^M c_{(0)}{}^N \\
&\quad = 0
\end{aligned} \tag{7.112}$$

The first term between the brackets vanishes because the $X_{LM}{}^K$ do not depend on the $\phi^i$. The second term is antisymmetric in $[MNL]$ since it is multiplied by the anticommuting ghost fields, and therefore it is equal to the left hand side of the modified Jacobi identity (4.89). Finally, the third term in (7.112) can accommodate the right hand side of the modified Jacobi identity, since it is proportional to $Y^K{}_{PQ}$. If we set

$$\widetilde{\boldsymbol{F}}^{\lceil PQ \rfloor}{}_{NML} c_{(0)}{}^L c_{(0)}{}^M c_{(0)}{}^N = \frac{1}{3} \delta_N{}^{\lceil P} X_{ML}{}^{Q \rfloor} c_{(0)}{}^L c_{(0)}{}^M c_{(0)}{}^N \,, \qquad (7.113)$$

we have shown that at antifield number 2 in the master equation, the modified Jacobi identity appears. This is clearly a consequence of the presence of the non-vanishing zero modes, that allow for an extra term that is proportional to $Y^K{}_{PQ}$. Likewise, several other relations can be found that are a consequence of the existence of the zero modes. Another example is the relation $Y^P{}_{RS} X_{PM}{}^N = 0$ that appears if one collects the terms proportional to $\phi^*_i c_{(0)}{}^{a_0} c_{(0)}{}^{b_0}$ in the master equation.

In the end, starting from the extended action and imposing the master equation, we are able to reproduce all the important relations that characterize the gauge structure of the embedding tensor formalism, and that were found before in the literature (e.g. in [61, 85, 135]).

## 7.5 Conclusions

Our results in this chapter extend previous work that has been done on the embedding tensor formalism in 4 space-time dimensions. It emphasizes the complicated form of the gauge algebra that was previously discussed in [85, 135], and tries to suggest a more concise description of the formalism via BV theory.

We started by calculating the full gauge algebra on the 1- and 2-form gauge fields. As these fields suffice to write down a gauge invariant action in 4 dimensions [61], no higher order form fields were considered. We argued that the algebra in the absence of any dynamics for the fields explicitly differs from the algebra in the presence of a gauge invariant action. In the latter case we showed that the algebra is open, i.e. only closes on-shell, whereas in the first case the algebra turned out to be closed. In both cases the algebra is soft since the 'structure constants' are functions of the fields. We also calculated the zero modes of the gauge transformations and proved that in both cases the algebra is higher-stage reducible. In principle we could conclude that the embedding tensor formalism is potentially even infinite stage reducible because the level at which the zero modes become independent cannot be determined. But as the discussion was very generic, we suggest that a case-by-case study of particular examples can bring more insight into this.

After having determined the relevant gauge structure tensors (generators, structure constants, zero modes, etc.) we used these tensors to construct a BV action. In this way all the features of the complicated gauge structure are captured by the BV framework and we conclude that this framework can be a convenient tool to further investigate the embedding tensor formalism.

An alternative approach would be to use the BV method of constructing stepwise an extended action, starting from a classical action. If we impose the $(S, S) = 0$ condition on the extended action, each term in the expansion must vanish separately and this gives rise to the known gauge structure relations (commutation relations, zero mode relations, Jacobi identities, etc.). Then the properties of the gauge structure tensors that we mentioned in earlier chapters follow from these relations.

It would be interesting to extend our results to arbitrary dimensions (especially in the cases where an action is known) and to study the gauge structure of the full tensor hierarchy, i.e., including higher order $p$-form fields. Also the reducibility of the theory remains an open question. As we said above, our discussion so far was very generic. Studying specific examples for which an explicit form of the projectors $\mathbb{P}$ is chosen can help us to get a better understanding of the level $L$ at which all zero modes become independent. Another way to study the reducibility of the theory is to do a dimensional analysis of the degrees of freedom of the theory. The total number of degrees of freedom of the theory depends strongly on the number of (higher stage) zero modes. By calculating this number explicitly, we can determine the level of reducibility $L$. This calculation might even be possible for general models and arbitrary space-time dimensions $D$.

Another subject to look at in the future is the quantization of the gauge theories that fall into the classification of the embedding tensor formalism. In this article we exploited the fact that the BV formalism provides a compact notation for the *classical* gauge structure of these theories. On the other hand, the BV formalism was originally designed as a method for the *quantization* of field theories. So, due to our reformulation of the embedding tensor formalism in terms of the BV formalism, we have now all the tools available for the quantization of generic gauged supergravities. In practice however, this might still be very hard to do.

APPENDIX A

# NOTATIONS AND CONVENTIONS

## Natural units

In particle physics, a widely adopted convention is to work in the system of *natural units*, in which

$$\hbar = c = 1\,. \tag{A.1}$$

This avoids having to keep track of factors of $\hbar$ and $c$. Only at the end is it necessary to convert back to more usual units. In conventional SI units $c$ has the value $c \sim 3 \times 10^8\,\mathrm{m\,s^{-1}}$. By choosing units such that $c = 1$, we are implying that our unit of length is numerically equal to our unit of time. In this sense, length and time are equivalent dimensions, $[L] = [T]$. Similarly, from the energy-momentum relation of special relativity $E^2 = \boldsymbol{p}^2 c^2 + m^2 c^4$ we see that the choice of $c = 1$ also implies that energy, mass and momentum all have equivalent dimensions. The numerical value of Planck's constant, on the other hand, is $\hbar \sim 6.6 \times 10^{-22}\mathrm{MeV\,s}$ and $[\hbar] = [M][L]^2[T]^{-1}$. Setting $\hbar = 1$ therefore relates units of $[M]$, $[L]$ and $[T]$. Since $[L]$ and $[T]$ are equivalent by our choice of $c = 1$, we can choose $[M]$ as the single independent dimension for the natural units:

$$[M] = [L]^{-1} = [T]^{-1}\,. \tag{A.2}$$

In the equations that are related to gravity, we will also set $\kappa^2 = 1$. In SI units, this quantity appears in the Einstein-Hilbert action,

$$S_{\rm EH} = \int {\rm d}^4x\, \sqrt{-g}\frac{R}{2\kappa^2}\,, \tag{A.3}$$

and it can be written in terms of Newton's constant or as the inverse of Planck's constant:

$$\kappa^2 = 8\pi G_N/c^4\,. \tag{A.4}$$

In high-energy physics, it is also not conventional to use SI units when treating electromagnetic phenomena or more general gauge theories. Instead, one chooses a coordinate system in which the field equations take their most convenient form. In particular one might make a choice such that the permeability $\mu_0$ and permittivity $\epsilon_0$ do not appear. In SI units we have $\mu_0 = 4\pi \times 10^{-7}\,{\rm kg\,m\,C}^{-2}$ and $\mu_0\epsilon_0 = 1/c^2$. If we set $\mu_0 = 1$ and $\epsilon_0 = 1$, this is consistent with our choice $c = 1$ and it relates the electrical quantities (Coulomb,...) to mechanical ones (mass, length, time). In particular, units of charge becomes dimensionless.

## Conventions

Throughout this work we use a space-time metric with signature $(- + ++)$. (Anti)symmetrization is done with weight one:

$$T_{[ab]} = \frac{1}{2}\left(T_{ab} - T_{ba}\right)\,, \qquad T_{(ab)} = \frac{1}{2}\left(T_{ab} + T_{ba}\right)\,. \tag{A.5}$$

The anticommuting Levi-Civita tensor is real:

$$\varepsilon_{0123} = 1, \quad \varepsilon^{0123} = -1\,, \tag{A.6}$$

and it satisfies the following contraction identity (in 4 dimensions):

$$\varepsilon_{a_1\ldots a_n b_1\ldots b_p}\varepsilon^{a_1\ldots a_n c_1\ldots c_p} = -n!p!\delta^{[c_1}_{[b_1}\ldots\delta^{c_p]}_{b_p]}\,. \tag{A.7}$$

Also for the local case, we define constant tensors:

$$\varepsilon_{\mu_1\ldots\mu_d} = e^{-1}e^{a_1}_{\mu_1}\ldots e^{a_d}_{\mu_1}\varepsilon_{a_1\ldots a_d}\,, \qquad \varepsilon^{\mu_1\ldots\mu_d} = e e^{\mu_1}_{a_1}\ldots e^{\mu_1}_{a_d}\varepsilon^{a_1\ldots a_d}\,. \tag{A.8}$$

Notice that indices on these tensors are therefore not brought up or down with the metric.

## Spinors

The Clifford algebra of gamma matrices takes the form

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}\,. \tag{A.9}$$

In 4 dimensions we define a fifth matrix $\gamma_5$ which is proportional to the product of all other gamma matrices,

$$\gamma_5 = \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3\,, \tag{A.10}$$

and $(\gamma_5)^2 = \mathbb{1}$.

Spinors will be denoted by letters from the Greek alphabet, common notations are $\lambda, \psi$ and $\chi$. These are anticommuting objects. In this text we mostly work with chiral spinors, i.e., spinors that are invariant under left- or right projections:

$$\lambda^{(L)} = \frac{1}{2}\,(\mathbb{1} + \gamma_5)\,\lambda^{(L)}\,, \qquad \lambda^{(R)} = \frac{1}{2}\,(\mathbb{1} - \gamma_5)\,\lambda^{(R)}\,. \tag{A.11}$$

In most cases it is not necessary to write spinor indices. If it is convenient to introduce them anyway, we use the following conventions. On basic spinors, the spinor index $\alpha, \beta, \ldots$ is always written as a lower index, i.e., $\lambda \to \lambda_\alpha$. Gamma matrices that work on the spinors get a lower and an upper index in the following order: $(\gamma_\mu)_\alpha{}^\beta$. For example one writes $\gamma_\mu\lambda$ as $(\gamma_\mu)_\alpha{}^\beta\lambda_\beta$. The Majorana conjugate of a spinor, $\bar{\lambda}$, get an upper spinor index i.e., $\lambda^\alpha$. Expressions that are bilinear in the spinors become

$$\bar{\lambda}\psi \;\to\; \lambda^\alpha\psi_\alpha\,, \qquad \bar{\lambda}\gamma_\mu\psi \;\to\; \lambda^\alpha(\gamma_\mu)_\alpha{}^\beta\psi_\beta\,, \text{ etc.} \tag{A.12}$$

APPENDIX B

# NEDERLANDSE SAMENVATTING

## B.1 Het nut van een effectieve theorie

In ons dagelijkse leven observeren we heel wat fysische fenomenen die op het eerste zicht niet beschreven kunnen worden binnen eenzelfde theoretisch kader. Denk maar aan de wetten van Kepler waarmee we de beweging van hemellichamen kunnen voorspellen, of thermodynamische relaties die ons helpen bij het bestuderen van vloeistoffen en gassen, of de wet van Ohm voor het analyseren van elektrische circuits, enzovoort. Hoewel een zekere systematiek aan de basis ligt voor het bestaan van elk van deze wetten, zijn er geen eenvoudige onderlinge relaties. Het is bijvoorbeeld niet duidelijk hoe dezelfde fysische principes tegelijkertijd verantwoordelijk kunnen zijn voor de ellipsvormige beweging van de aarde rond de zon en de beschrijving van de druk in een gas. Toen deze grote hoeveelheid aan schijnbaar ongerelateerde fysische wetten in de loop van de eeuwen aan het licht is gekomen, werden ze daarom door fysici nooit aanzien als het eindpunt van hun onderzoek. Integendeel, het was voor veel wetenschappers een drijfveer om op zoek te gaan naar een meer fundamentele beschrijving, naar onderliggende, microscopische verklaringen en naar een theorie die meer structuur zou geven aan de verscheidenheid rondom ons. Gedurende de laatste 300 jaar heeft deze zoektocht stilaan vorm gekregen, en in de 20ste eeuw is ze uitgemond in de constructie van twee “fundamentele” theorieën, namelijk het Standaardmodel en algemene relativiteitstheorie. Deze twee theorieën kunnen (in principe) de fysische verscheidenheid rondom ons verklaren aan de hand van een klein aantal elementaire deeltjes en vier soorten interacties (gravitatie, elektromagnetisme, de sterke en zwakke interactie).

Ondanks deze grote vooruitgang zijn we er toch niet in geslaagd om, gebruik makend van onze kennis over de elementaire deeltjes en interacties, alle fysische fenomenen te beschrijven die we rondom ons waarnemen. Het is bijvoorbeeld niet mogelijk om de interacties tussen een willekeurig aantal protonen en neutronen te beschrijven, vertrekkend van ons wiskundig model voor de sterke wisselwerking tussen quarks. Bovendien kunnen we ons de vraag stellen of we wel altijd genteresseerd zijn in de volledige microscopische beschrijving. Het is toch ruimschoots voldoende om een vloeistof te beschrijven aan de hand van zijn dichtheid, temperatuur, druk, enzovoort, in plaats van alle eigenschappen van alle moleculen in de vloeistof te kennen? Daarom hebben fysici altijd gewerkt met zogenaamde "effectieve theorieën". Dat zijn modellen die blind zijn voor (een deel van) de details van de onderliggende microscopische structuur. Alle empirische wetten die we hierboven hebben vermeld vallen binnen deze categorie. Bijvoorbeeld, de gebruikte grootheden in thermodynamica zijn druk, temperatuur, dichtheid, en niet de posities, momenta en massa's van alle moleculen in een vloeistof of gas. Wat gebeurt er als we deze redenering doortrekken tot het Standaardmodel (SM) en Einsteins theorie voor gravitatie (GR)? Zijn deze theorieën echt fundamenteel, of zijn het ook effectieve beschrijvingen van een (ongekende) onderliggende werkelijkheid? Het is natuurlijk zo dat we nog nooit experimentele afwijkingen hebben waargenomen ten opzichte van de theoretische voorspellingen van het SM en GR. Binnen de huidige experimentele grenzen mogen we dus besluiten dat zowel het SM als GR een accurate beschrijving geven van de werkelijkheid. Maar dit sluit niet uit dat er een regime bestaat waar het SM en GR toch hun geldigheid verliezen, maar waar we momenteel met onze experimenten geen toegang toe hebben. Inderdaad, er zijn zeer sterke theoretische aanwijzigen dat dit scenario zich voordoet bij energieën in de buurt van de Planck schaal ($\sim 10^{18}$GeV), waar een nieuwe fysische theorie noodzakelijk is. Deze nieuwe theorie wordt door fysici ook wel omschreven als "kwantumgravitatie".

## B.2 Op weg naar een kwantumtheorie voor gravitatie

Het Standaardmodel geeft ons een geünificeerde beschrijving van de elektromagnetische, zwakke en sterke wisselwerkingen binnen het kader van een kwantumveldentheorie. De belangrijkste ingrediënten van zo'n theorie zijn de fysische velden, zoals het elektrische en magnetische veld uit Maxwells beschrijving van elektromagnetisme. Na kwantisatie beschrijft elk van deze velden een elementair puntdeeltje, en interageren de verschillende deeltjes via het uitzenden en absorberen van zogenaamde krachtdragende deeltjes. De elektromagnetische interactie tussen elektronen, bijvoorbeeld, wordt gemedieerd via het uitwisselen van fotonen.

Om efficiënt met zulke emissie- en absorptieprocessen om te kunnen gaan, heeft Feynman een elegante grafische voorstelling voor deze processen ontwikkeld. Het belangrijkste onderdeel van Feynmans voorstelling is de lokale interactievertex, die het uiteenvallen van een deeltje in een of meerdere nieuwe deeltjes beschrijft. Aan iedere vertex associeert men een object (een getal of een matrix) dat proportioneel is met de sterkte van de interactie tussen de deeltjes die in de vertex samenkomen. Bijvoorbeeld, wanneer twee elektronen interageren met een foton, is de overeenkomstige vertex evenredig met de elektrische lading, $e$. In het algemeen is deze evenredigheidsconstante –die we ook de koppelingsconstante noemen– enkel afhankelijk van het type interactie en de interactie energie.

Wanneer de exacte vorm van iedere vertex gekend is, geeft kwantumveldentheorie ons een efficiënt algoritme om de waarschijnlijkheid van een willekeurig fysisch proces te berekenen. Het resultaat wordt bekomen als volgt. Eerst construeert men alle mogelijke diagrammen die bestaan uit het aaneenschakelen van vertices, zondanig dat de fysische begin- en eindtoestand overeenkomen met die van het proces dat men wil bestuderen. Aan ieder diagram kan dan een waarschijnlijkheid worden verbonden die evenredig is met een macht van de koppelingsconstantes. In het algemeen neemt deze macht toe naarmate er meer vertices in het diagram voorkomen. Zo produceert kwantumveldentheorie een oneindige machtreeks voor de uitkomst van ieder fysisch proces, met de koppelingsconstantes als expansieparameters. Als de koppelingsconstantes klein genoeg zijn, zoals bijvoorbeeld het geval is voor elektromagnetisme bij lage energieën, dan kan de machtreeks afgekapt worden na een eindig aantal termen, zodanig dat het resultaat nog steeds een goede benadering vormt voor de exacte uitkomst van het fysische proces. Zulke "perturbatieve berekeningen" zijn vaak de enige manier waarop we met kwantumvelden theorie tot een kwantitatieve voorspelling kunnen komen.

Maar behalve het succes van de perturbatieve expansie leidt een beschrijving in termen van kwantumvelden ook tot fundamentele problemen die gerelateerd zijn aan de noodzakelijke introductie van kwantumgravitatie. Eerst en vooral merken we op dat een rechttoe rechtaan toepassing van de hierboven beschreven procedure leidt tot oneindige resultaten voor de berekening van bepaalde fysische processen. Het is niet enkel de volledige machtreeks die divergeert, maar oneindigheden komen voor bij verschillende ordes in de expansie. Een oplossing voor dit probleem werd eerst gevonden voor elektromagnetisme, en het leidde tot de ontwikkeling van een ingewikkelde procedure die men renormalisatie heeft genoemd. In een notendop absorbeert men tijdens deze procedure alle oneindigheden via de herdefinitie van onfysische parameters. Een theorie is renormaliseerbaar als alle oneindigheden door de herdefinitie van een eindig aantal parameters weggewerkt kunnen worden. In de jaren 70 kon men zo aantonen dat het Standaardmodel voldoet aan deze eis, met als gevolg dat deze theorie perfect eindige voorspellingen kan doen voor alle fysische processen in termen van een eindig aantal onbekende parameters (18 in totaal), die experimenteel bepaald moeten worden.

Een kwantumveldentheorie voor gravitatie, aan de andere kant, is niet renormaliseerbaar. De reden is dat de divergenties in een gravitatietheorie veel strenger zijn als voor de andere interacties. Dit komt omdat de interactiesterkte voor processen met een graviton schalen met het kwadraat van de energie, in vergelijking met een logaritmische afhankelijkheid voor de andere interacties. De persistente aanwezigheid van deze oneindigheden is op z'n minst vervelend te noemen. Meer nog, men kan geen exacte voorspellingen doen voor willekeurige fysische processen in termen van een eindig aantal meetbare parameters. Maar dat betekent niet dat algemene relativiteitstheorie fout zou zijn, we kunnen enkel besluiten dat we de theorie proberen te gebruiken voor een fysisch regime (namelijk een zeer hoge energie) waar ze niet meer geldig is. De correcte beschrijving van fysische processen bij deze hoge energie wordt daarom gegeven door een nieuwe theorie voor kwantumgravitatie.

## B.3 Leven we in een supersymmetrische wereld?

Gedurende de afgelopen 30 jaar hebben fysici een zeer goed beeld gekregen van hoe een dergelijke theorie voor kwantumgravitatie eruit moet zien. We hebben zelfs een veelbelovende kandidaat gevonden, namelijk supersnaartheorie. In deze nieuwe beschrijving zijn elementaire deeltjes geen geïdealiseerde punten meer, maar trillende snaartjes of membranen met meerdere dimensies. Snaren interageren met elkaar via opsplitsen en aaneensmelten, met een interactiesterkte die wordt bepaald door de snaarkoppelingsconstante $g_s$. Aangezien de interacties door de eindige snaarlengte uitgesmeerd zijn in de ruimtetijd, is snaartheorie eindig order per orde in de perturbatie. Daarom heeft ze ook niet af te rekenen met de oneindigheden die in een gewone kwantumvelden theorie wel de kop op steken.

Echter, de hoop om ooit snaartjes in experimenten waar te nemen is zo goed als onbestaande. Daarvoor zouden we namelijk versnellers nodig hebben die energieën kunnen bereiken van de Planck schaal, en dat is zo'n $10^{15}$ maal krachtiger als wat tegenwoordig met de LHC of Tevatron mogelijk is. Men kan zich dus afvragen of we ooit snaartheorie experimenteel zullen kunnen testen. Toch is het antwoord op deze vraag waarschijnlijk positief, aangezien een aantal karakteristieke eigenschappen van snaartheorie zich ook bij lage energie manifesteren. Een van deze eigenschappen is een eigenaardige symmetrie van de ruimtetijd, die we supersymmetrie noemen. Het bijzondere aan supersymmetrie is dat ze het bestaan van nieuwe deeltjes voorspelt, namelijk de zogenaamde superpartners van de al gekende bosonen en fermionen. Zo is ieder boson aan een fermion gerelateerd, en vice versa. We verwachten dus nieuwe bosonische partners van de quarks te observeren (squarks genoemd), of een fermionische partner voor het graviton (een gravitiono), enzovoort. De reden waarom nog geen harde experimentele bewijzen gevonden zijn voor het bestaan van deze supersymmetrische partners, is omdat supersymmetrie gebroken is bij lage energie, en deze deeltjes daarom een massa krijgen die tot vandaag niet waarneembaar is in experimenten. Toch hopen fysici dat de LHC hier binnenkort verandering in zal brengen. Deze versneller zal energieën bereiken van de grootte-orde TeV/proton, wat volgens theoretische modellen precies overeenkomt met het regime waar superpartners “zichtbaar” worden.

Onafhankelijk van snaartheorie werd supersymmetrie ook in normale vierdimensionale veldentheorieën geïmplementeerd. Dit werd voor het eerst gedaan door Wess en Zumino in 1974 [5, 6], toen zij een supersymmetrische uitbreiding vonden voor bestaande theorieën, via de introductie van de correcte superpartners. Niet lang hierna ontdekten Ferrara, Freedman en Van Nieuwenhuizen ook een manier om gravitatie aan deze constructie toe te voegen [7]. Deze supersymmetrische uitbreidingen voor gravitatie –of supergravitaties– werden zeer enthousiast onthaald door theoretici, aangezien ze een voldoende restrictief kader vormen voor de beschrijving van zowel de interacties uit het Standaardmodel als voor gravitatie. Fysici gingen zelfs zover om supergravitatie te bestempelen als de gezochte theorie voor kwantumgravitatie. En daar hadden ze in het begin goede redenen voor. Het bleek namelijk dat als men de eerste ordes in de perturbatieve expansie voor supergravitatie berekent, alle oneindigheden op een miraculeuze wijze verdwijnen door nieuwe contributies die voortkomen uit de propagatie van (tot voorheen onbekende) superdeeltjes. Daardoor leek het alsof supergravitatie eindig is, ten minste tot op tweede orde in de expansie. Maar jammer genoeg toonden meer ingewikkelde berekeningen later aan dat voor hogere ordes de oneindigheden toch terug de kop op steken, waarmee ook de hoop op een consistente kwantumgravitatie verdween.

Deze tegenslag betekende echter niet het einde van onze interesse in supergravitatie. In tegendeel, wanneer supersnaartheorie vanaf 1984 aan populariteit won, veranderde ook onze kijk op supergravitatie. Inderdaad, supergravitatie geeft een effectieve beschrijving van snaartheorie in de lange-afstands limiet, waar alle effecten ten gevolge van een eindige stringlengte verwaarloosd kunnen worden. Met andere woorden, als men het spectrum van snaartheorie beschouwt, vindt men een massaloze spin-2 excitatie die overeenkomt met het graviton, maar ook massaloze spin-1 velden, scalaire velden, en antisymmetrische $p$-vormen in het algemeen. De dynamica van al deze massaloze velden wordt beschreven door een supergravitatie-actie die een kinetische term bevat voor elk propagerend deeltje (inclusief een Einstein-Hilbert term voor het gravition), maar ook spin-1 interacties zoals in het Standaardmodel, en verschillende andere koppelingen die sterk worden beperkt door diffeomorfisme en supersymmetrie invariantie. Precies door deze connectie tussen snaartheorie en supergravitatie wordt vandaag de dag heel wat vooruitgang geboekt in snaartheorie via het bestuderen van de overeenkomstige supergravitatie.

## B.4 Snaren in realiteit

Er is echter n aspect dat we tot nog toe over het hoofd hebben gezien, namelijk het feit dat een consistentie supersnaartheorie onvermijdelijk in tien ruimtetijds dimensies geformuleerd moet worden. Deze eigenschap leidt tot het conceptuele probleem of (en hoe) we snaartheorie kunnen linken aan een observeerbare, vierdimensionale theorie. Een goed begrip van hoe deze connectie werkt is van groot belang als we ooit de beschrijvende en voorspellende kracht van snaartheorie willen uitbuiten. Zoals we onmiddellijk zullen zien is er nog steeds geen eenduidige oplossing voor dit probleem. De standaardprocedure om met de zes extra dimensies in snaartheorie om te gaan, is via de studie van een speciale klasse van oplossingen. Deze oplossingen veronderstellen een ruimtetijdsgeometrie van de vorm vorm $\mathbb{R}^{1,3} \times K^6$ met $\mathbb{R}^{1,3}$ de vierdimensionale Minkowksi ruimte en $K^6$ een compacte interne ruimte. Als de compacte dimensies klein genoeg zijn, hebben de verschillende massaloze deeltjes die snaartheorie voorspelt een effectieve beschrijving in termen van een *vierdimensionale* supergravitatie theorie. Deze procedure wordt ook wel “dimensionele reductie” genoemd, en ze geeft een zeer natuurlijke beschrijving voor het feit dat we slechts een vierdimensionale wereld waarnemen. Maar deze aanvaardbare methode geeft ook aanleiding tot een van de belangrijkste problemen waar we vandaag de dag een antwoord op proberen te vinden. Het is namelijk zo dat in de loop van de jaren een enorm aantal consistentie reducties werd geconstrueerd, die elk aanleiding geven tot een vierdimensionale theorie waarvan de eigenschappen afhangen van bijvoorbeeld de vorm van $K^6$. Er bestaat echter nog geen mechanisme dat uit deze enorme verzameling mogelijkheden de correcte oplossing selecteert die ons universum beschrijft. Daarom is het moeilijk, zoniet onmogelijk, om fysische voorspellingen te doen.

We moeten ons dus afvragen of er geen andere manier bestaat waarop we vooruitgang kunnen boeken. Is het bijvoorbeeld nuttig om alle gekende compactificaties te testen op overeenkomsten met de realiteit? Deze procedure is zeker niet eenvoudig en zal niet onmiddellijk tot unieke antwoorden leiden. Toch hebben snaartheoreten ook hier al heel wat geleerd. Zo vertonen bepaalde klassen van compactificatiemodellen sterke overeenkomsten met de realiteit. Men weet bijvoorbeeld hoe de drie quark en lepton generaties in het Standaardmodel gelinkt kunnen worden aan de geometrie van de interne ruimte. Ander eigenschappen van de elementaire deeltjes, zoals hun massa of lading, kunnen ook in verband worden gebracht met bepaalde eigenschappen van $K^6$, enzovoort. Ondanks het feit dat de details van deze constructies moeilijk te begrijpen zijn, is het toch duidelijk dat snaartheorie een kader biedt waarbinnen we bepaalde vragen kunnen beantwoorden waarop de gekende theorieën zoals het Standaardmodel zeker geen antwoord bieden. Niettegenstaande deze positieve aspecten zijn we nog ver verwijderd van een volledige oplossing, zeker zolang er geen precies mechanisme wordt gevonden waarmee we de “juiste” snaarconfiguratie kunnen selecteren uit de oneindige verzameling van mogelijkheden.

Zolang deze selectieregels niet opgesteld zijn, worden we dus eigenlijk gedwongen om naar generieke, in plaats van specifieke, eigenschappen te kijken van een wereld vol snaren. In deze context duidt generiek op de universele eigenschappen die snaartheorie ons oplegt, onafhankelijk van de compactificatie die men beschouwt. Twee karakteristieke eigenschappen zijn we reeds vroeger in deze tekst tegen gekomen, namelijk (i) het feit dat snaartheorie het bestaan van een graviton voorspelt met (lange-afstands) interacties die overeenkomen met die van algemene relativiteitstheorie, and (ii) het bestaan van supersymmetrie als een nieuwe symmetrie voor de ruimtetijd. In vier dimensies worden deze eigenschappen mooi gecombineerd binnen een theorie van pure supergravitatie, die naast het graviton ook de superpartners (gravitini, scalairen, enz.) beschrijft. Behalve deze basisingrediënten zijn er twee andere generieke aspecten van snaartheorie waar we zeker wat meer aandacht aan moeten schenken.

- Ten eerste voorspelt snaartheorie de aanwezigheid van extra materievelden, bovenop het universele graviton, de gravitini, enzovoort. Bijvoorbeeld, in vier dimensies kan de minimale veldinhoud uitgebreid worden met extra spin-1/spin-$\frac{1}{2}$ en spin-$\frac{1}{2}$/spin-0 doubletten. In principe hangen de eigenschappen van deze extra velden af van de precieze details van de interne geometrie, maar ze kunnen ook in een onafhankelijke supergravitatiecontext bestudeerd worden. Het voordeel van deze “materie-gekoppelde supergravitaties” is dat het aantal deeltjes nauwkeurig in overeenstemming kan worden gebracht met het waargenomen spectrum van elementaire deeltjes.
- Ten tweede brengen snaartheorie compactificaties in het algemeen ook interacties voort tussen spin-1 velden en andere materiedeeltjes, met een structuur die zeer dicht aanleunt bij die van de interacties in het Standaardmodel. Deze eigenschap biedt nieuwe en interessante perspectieven. Eerst en vooral verklaart dit waarom alle fundamentele interacties dezelfde universele structuur hebben. Ten tweede kunnen de Standaardmodelinteracties nu ingebed worden in een supergravitatie theorie. Tot slot biedt dit ook de mogelijkheid voor nieuwe, zwakke interacties die tot nu toe niet gedetecteerd werden. Deze nieuwe interacties kunnen bepaalde hypothetische processen verklaren die niet toegelaten zijn door het Standaardmodel, zoals een traag proton verval of transmutaties tussen verschillende quarkcombinaties. De supergravitatietheorieën die zulke interacties beschrijven noemen we “geijkte supergravitaties”.

Met deze opsomming van karakteristiek eigenschappen hopen we de lezer ervan overtuigd te hebben dat het de moeite waard is om vierdimensionale supergravitatietheorieën te bestuderen. Aangezien deze theorieën in de toekomst als toetssteen zullen worden gebruikt bij het verifiëren van snaartheorie, is het belangrijk om nu reeds hun structuur te begrijpen, en om een zo volledig mogelijke classificatie te maken van alle mogelijke materiekoppelingen en ijkingen. Uiteindelijk kan deze classificatie –in combinatie met experimentele waarnemingen– dan leiden to de constructie van realistische modellen. Dit is tegelijkertijd ook de bredere context waarbinnen het werk in deze thesis moet worden geplaatst.

## B.5 Inhoud van deze thesis

In deze tekst gaat onze interesse hoofdzakelijk uit naar vierdimensionale supergravitaties met een minimale hoeveelheid supersymmetrie (wat conventioneel wordt genoteerd met $\mathcal{N} = 1$) en algemene koppelingen aan vector- en chirale multipletten. De structuur van deze theorieën wordt stap voor stap geïntroduceerd in de eerste drie hoofdstukken. Hoofdstuk 2 bevat een eenvoudige inleiding tot algemene ijktheorieën, met als voornaamste voorbeeld het Standaardmodel. In hoofdstuk 3 verduidelijken we de rol van (lokale) ruimtetijdssymmetrieën en introduceren we gravitatie, supersymmetrie en supergravitatie. Tot slot worden al deze resultaten samengevoegd in hoofdstuk 4, wat leidt tot een eerste kennismaking met geijkte supergravitaties. Deze theorieën werden vroeger reeds bestudeerd in de literatuur [8,9], maar er zijn nog heel wat onopgeloste vragen. De belangrijkste onvolkomenheid is het gebrek aan een complete opsomming van alle mogelijke geijkte versies. In ons werk zullen we daarom een belangrijke stap zetten in deze richting, en ontrafelen we de ingewikkelde structuur van meer algemene geijkte supergravitaties.

In hoofdstuk 5 bestuderen we de ijking van een meer algemene subgroep van de elektromagnetische dualiteitstransformaties. In het bezonder kijken we naar transformaties die op de scalaire velden werken met een lokale shift-symmetrie. In het algemeen is de originele supergravitatie actie niet invariant onder deze lokale transformaties, aangezien de Peccei-Quinn term op een niet-triviale manier transformeert. Om invariantie van de actie te herstellen moeten we daarom veralgemeende Chern-Simons termen toevoegen.[1] Deze termen zijn kubisch en kwartisch in de ijkvelden, en hun transformaties vallen precies weg tegenover de bijdragen van de Peccei-Quinn term. Het blijkt echter dat niet alle shift-transformaties op deze manier geijkt kunnen worden, aangezien ijk- en supersymmetrie invariantie bepaalde voorwaarden opleggen. Toch zullen we in het tweede deel van hoofdstuk 5 aantonen dat deze condities verder afgezwakt kunnen worden. In dit geval is de klassieke actie niet meer ijkinvariant en supersymmetrisch, maar de niet-triviale variaties kunnen wel worden gebruikt

[1] Veralgemeende Chern-Simons termen werden vroeger reeds geïntroduceerd voor theorieën met uitgebreide supersymmetrie, maar wij hebben aangetoond dat ze ook aanwezig kunnen zijn in het geval $\mathcal{N} = 1$.

om ongewenste kwantumanomalieën onschadelijk te maken. Dit wordt ook wel het Green-Schwarz mechanisme genoemd. We zullen de noodzakelijke en voldoende voorwaarden bespreken die leiden tot de afwezigheid van zowel ijk- als supersymmetrie- en gemixte anomalieën.

In hoofdstuk 6 werken we voort rond hetzelfde thema, en gaan we op zoek naar de structuur van de meest algemene ijkingen. Hiervoor zullen we gebruik maken van een manifest elektromagnetisch invariante formulering in termen van het embedding tensor formalisme. Veralgemeende Chern-Simons termen zijn generiek aanwezig in dit formalisme, net als de (niet-afgezwakte) condities op de shift-transformaties in hoofdstuk 5. Deze condities zijn nu een deel van een meer algemene relatie die we de representatieconstraint noemen. Het is deze beperking die mee bepaalt welke ijkingen mogelijk zijn. Onze interesse in hoofdstuk 6 gaat echter vooral uit naar een afgezwakte versie van deze beperking, in dezelfde lijn als de zwakkere conditie in hoofdstuk 5. Het blijkt dat de aangepaste representatieconditie opnieuw toegelaten is als we tegelijkertijd ijkanomalieën in rekening brengen. Zo krijgt de representatieconditie een zinvolle fysische betekenis: in z'n originele vorm beschrijft ze de afwezigheid van kwantumanomalieën.

Tot slot keren we in hoofdstuk 7 terug naar het embedding tensor formalism zonder anomalieën, en bestuderen we in meer detail de ingewikkelde ijkstructuur van dit formalisme. Het blijkt immers dat naast de vectorvelden ook antisymmetrische 2-vormen en verschillende "types" ijktransformaties aanwezig zijn. We tonen aan dat de ijkalgebra geassocieerd aan deze velden en transformaties in het algemeen een open, soft en reducibele algebra is. Open algebra's hebben een commutator van twee ijktransformaties die enkel sluit modulo termen die proportioneel zijn met de veldvergelijkingen. Soft algebra's bezitten structuurconstanten die afhangen van de velden. Reducible algebra's, tot slot, hebben ijktransformaties die niet onafhankelijk zijn. De coëfficiënten die de afhankelijkheid bepalen worden ook wel zero-modes genoemd. In het geval van het embedding tensor formalisme blijkt dat ook de zero-modes niet onafhankelijk zijn, waardoor we met een hogere-orde reducibele algebra te maken hebben. Het is niet helemaal duidelijk of deze hiërarchie van hogere-orde zero-modes afbreekt na een eindig aantal stappen, of dat we met een oneindig reducibele algebra te maken hebben. Naast een beter begrip van de ijkstructuur van het embedding tensor formalisme is het tweede doel van ons werk in hoofdstuk 7 om een bondigere beschrijving te vinden voor deze ingewikkelde structuur. Daarom zullen we het embedding tensor formalisme herformuleren in termen van het klassieke Batalin-Vilkovisky (of veld-antiveld) formalisme. We vinden een zeer compacte beschrijving voor de verschillende objecten die de ijkstructuur bepalen (zoals ijkgeneratoren, structuurfuncties, zero-modes, Jacobi identiteiten, enz.) in termen van n "mastervergelijking". Bovendien hebben we met het Batalin-Vilkovisky formalisme nu ook alle tools ter beschikking voor de kwantisatie van algemene ijktheorieën.

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